Institut für
Volkswirtschaftslehre
und Statistik

No. 572-99

Minimax strategies in survey sampling

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Beiträge zur angewandten Wirtschaftsforschung

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Abstract

The risk of a sampling strategy is a function on the parameter space, which is the set of all vectors composed of possible values of the variable of interest. It seems natural to ask for a minimax strategy, minimizing the maximal risk.

So far answers have been provided for completely symmetric parameter spaces. Results available for more general spaces refer to sample size 1 or to large sample sizes allowing for asymptotic approximation.

In the present paper we consider arbitrary sample sizes, derive a lower bound for the maximal risk under very weak conditions and obtain minimax strategies for a large class of parameter spaces. Our results do not apply to parameter spaces with strong deviations from symmetry. For such spaces a minimax strategy will prescribe to consider only a small number of samples and takes a non-random and purposive character.

AMS classification: primary 62D05, secondary 90D45

Keywords: Survey sampling; Maximal risk; Minimax strategies; Linear regression models

1. Introduction

Consider a population of units $1, 2, \ldots N$ and associated values $y_1, y_2, \ldots y_N$ of a characteristic of interest. The parameter (vector) $\underline{y} = (y_1, y_2, \ldots y_N)'$, and especially the parameter sum $y = y_1 + y_2 + \ldots + y_N$ are unknown to us. So we select a sample s of size n, i.e. an element of

$$S = \{ s : s \subset \{1,2,\ldots N\}, |\, s \,| = n \},$$

choose weights $a_{si}, i \in s$, ascertain the values $y_i, i \in s$, and estimate y by

$$\sum_{i \in s} a_{si}\, y_i \; .$$

A sample may be selected randomly. Let p_s be the probability of selecting $s \in S$; then $p : s \to p_s$ is called sampling design. An estimator is a function t assigning a real value

$$t\,(s, \underline{y}) = \sum_{i \in s} a_{si}\, y_i$$

to each pair of a sample $s \in S$ and a parameter $\underline{y}$.

$$R(\underline{y}; p, t) = \sum_{s} p_s \left[t(s, \underline{y}) - y \right]^2$$

is the risk of the strategy (p, t), p a design and t an estimator.

The strategy we use should reflect our prior knowledge. The set of a-priori possible parameters is called parameter space Θ. Several authors have considered the space

$$T^{(1)} = \left\{ \underline{y} \in \Re^N : \sum \left(y_i - \bar{y}\right)^2 \le c^2 \right\}$$

with $\bar{y} = y / N$ and $c \ne 0$; see Bickel and Lehmann (1981), Gabler (1990). Stenger and Gabler (1996) discuss, more generally,

$$Q^{(2)} = \left\{ \underline{y} \in \Re^N : \sum \sum d_{ij} \left(y_i - \bar{y}\right)\left(y_j - \bar{y}\right) \le c^2 \right\}$$

with (d_{ij}) a positive definite $N \times N$ matrix. Usually, values $x_1, x_2, \ldots x_N > 0$ of an auxiliary variable related to the variable of interest are available and, especially, Θ may depend on $\underline{x} = (x_1, x_2, \ldots x_N)'$. An example is

$$Q^{(3)} = \left\{ \underline{y} \in \Re^N : \sum \left(\frac{\bar{x}}{x_i} \right) \left(y_i - \frac{y}{x} x_i \right)^2 \leq c^2 \right\}$$

with $x = x_1 + x_2 + \ldots + x_N$ and $\bar{x} = x / N$. See Stenger (1989) and Gabler (1990). We refer to Cheng and Li (1983, 1987) for further examples.

In the present paper we consider

$$Q = \left\{ \underline{y} \in \Re^N : \underline{y}' U \underline{y} \leq c^2 \right\} \tag{1}$$

where U is non-negative definite of rank N-1 with

$$U \underline{x} = \underline{0} ,$$

$\underline{0} = (0, 0, \ldots 0)' \in \Re^N$. In a subsequent paper we will give a detailed justification of this approach. Presently we confine ourselves to note that the spaces $\Theta^{(1)}, \Theta^{(2)}$ and $\Theta^{(3)}$, discussed in the literature, are special cases of Θ. Additional comments are given in section 7.

The condition

$$x = 1$$

is not restrictive and will be assumed throughout the paper. Obviously,

$$\sum\left(y_i-\bar{y}\right)^2=\underline{y}'\,U^{(1)}\,\underline{y}$$

$$\sum\sum d_{ij}\left(y_i-\bar{y}\right)\left(y_j-\bar{y}\right)=\underline{y}'\,U^{(2)}\,\underline{y}$$

$$\frac{1}{N}\sum\frac{1}{x_i}\left(y_i-yx_i\right)^2=\underline{y}'\,U^{(3)}\underline{y}$$

with

$$U^{(1)}=I-\frac{1}{N}\underline{1}\underline{1}'$$
$$U^{(2)}=\left(I-\frac{1}{N}\underline{1}\underline{1}'\right)\left(d_{ij}\right)\left(I-\frac{1}{N}\underline{1}\underline{1}'\right)$$
$$U^{(3)}=\left(I-\underline{1}\,\underline{x}'\right)\cdot\frac{1}{N}\operatorname{diag}^{-1}\left(\underline{x}\right)\cdot\left(I-\underline{x}\,\underline{1}'\right)$$
$$=\frac{1}{N}\operatorname{diag}^{-1}\left(\underline{x}\right)-\frac{1}{N}\underline{1}\underline{1}'$$

and

$$U^{(i)}\,\underline{x}=\underline{0}$$

in all cases, with $\underline{x}=\underline{1}/N$ for $i=1,2$. Here and subsequently, $\underline{1}$ is the N-vector with all components equal to 1; I is the $N\times N$ identity matrix and diag($\underline{x}$) the diagonal matrix D with $d_{ii}=x_i$ for $i=1,2,\ldots N$.

2. Main results

Define

$$r(p,t)=\sup_{\underline{y}\in T} R\left(\underline{y};p,t\right)$$

A strategy $\left(\overset{*}{p},\overset{*}{t}\right)$ is minimax if

$$r\left(\overset{*}{p},\overset{*}{t}\right)=\min_{(p,\,t)} r(p,t)<\infty$$

For $\Theta=\Theta^{(1)}$ we have

$$\min_{(p,t)} r(p,t)=\frac{N}{n}\frac{N-n}{N-1}c^2$$

$$=r\left(\overset{*}{p},\overset{*}{t}\right)$$

where $\overset{*}{p}$ denotes simple random sampling without replacement, i.e. $\overset{*}{p}_s=1\Big/\binom{N}{n}$ for all $s\in S$,

and

$$\overset{*}{t}(s,\underline{y})=\frac{N}{n}\sum_{i\in s} y_i$$

is the expansion estimator. See e.g. Stenger (1979), Bickel and Lehmann (1981), Gabler (1990). Hence, a minimax strategy is available in case

$$U=U^{(1)}$$
$$\underline{x}=\underline{1}/N.$$

Stenger and Gabler (1996) derive a minimax strategy for

$$U \text{ close to } U^{(1)}$$
$$\underline{x}=\underline{1}/N.$$

In the present paper we assume

$$U \text{ close to } U^{(1)}$$
$$\underline{x} \text{ close to } \underline{1}/N$$

and show the following:

Let z_o be the unique solution of

$$|N-2n|z = \sum_1^N \sqrt{z^2 - x_i}$$

and define κ = sgn(N-2n) and for $i = 1, 2, \ldots N$

$$d_i = \frac{z_o + \kappa\sqrt{z_o^2 - x_i}}{x_i} \quad .$$

Then, an estimator $\overset{*}{t}$ and a design $\overset{*}{p}$ exists such that $\left(\overset{*}{p}, \overset{*}{t}\right)$ is minimax where $\overset{*}{t}$ is defined by

$$\overset{*}{t}(s, \underline{y}) = \sum_{i \in s} a_{si}^{*} y_i = \frac{\sum_{i \in s} d_i y_i}{\sum_{i \in s} d_i x_i} \quad .$$

An explicit formula for the design $\overset{*}{p}$ will be given in Theorem 3.

Defining $\alpha_i = d_i x_i$, i = 1, ... N, $\overset{*}{t}(s, \underline{y})$ can be written as a Hansen Hurwitz type estimator

$$\overset{*}{t}(s, \underline{y}) = \frac{\sum_{i \in s} \alpha_i \frac{y_i}{x_i}}{\sum_{i \in s} \alpha_i}$$

Note that the α_i´s do not depend on U, while the design $\overset{*}{p}$ does. The α_i´s and $\overset{*}{p}$ are free of c.

We give an example. Let N=3, n=2 and $2x_i < 1$ for i = 1, 2, 3. Define

$$\alpha_i = \frac{1}{(1-2x_i)} \sqrt{0.5 \cdot \prod (1-2x_k)} \quad \text{for i = 1, 2, 3}$$

and for $s = \{i, j\}$, $i \neq j$

$$\overset{*}{t}(s, \underline{y}) = \frac{\frac{1}{(1-2x_i)}}{\frac{1}{(1-2x_i)} + \frac{1}{(1-2x_j)}} \frac{y_i}{x_i} + \frac{\frac{1}{(1-2x_j)}}{\frac{1}{(1-2x_i)} + \frac{1}{(1-2x_j)}} \frac{y_j}{x_j} \quad .$$

$$\overset{*}{p}_s = (1 - x_i - x_j)\left(\frac{2x_i}{1-2x_i} + \frac{2x_j}{1-2x_j} - \sum \frac{x_k}{1-2x_k}\right) .$$

If $U = \Theta^{(3)}$ and $\overset{*}{p}_s$ is nonnegative for all samples s, $\left(\overset{*}{p}, \overset{*}{t}\right)$ is the minimax strategy. The risk of $\left(\overset{*}{p}, \overset{*}{t}\right)$ at $\underline{y}$ is

$$R(\underline{y}; \overset{*}{p}, \overset{*}{t}) = \frac{1}{\frac{1}{3}\sum\left(\frac{x_i(1-2x_i)^2}{\prod(1-2x_k)} + x_i\right)} \underline{y}' U^{(3)} \underline{y}$$

3. Interpretation of the main results: game and regression theory

Consider the following 2-person 0-sum game:

Player I, called Nature, selects $\underline{y} \in \Theta$, Θ *defined by* (1). *Independently, Player II, called Statistician, selects* $s \in S$ *and* $a_{si}, i \in s$ *and has to pay*

$$\left(\sum a_{si} y_i - y\right)^2 .$$

Let $\underline{a}_s^0$ be the N-vector with

$$i - \text{th component} = \begin{cases} a_{si} & \text{if } i \in s \\ 0 & \text{otherwise} \end{cases}$$

Then, the pay-off

$$\left[\left(\underline{a}_s^0 - \underline{1}\right)' \underline{y}\right]^2 = \left(\underline{a}_s^0 - \underline{1}\right)' \underline{y}\,\underline{y}'\left(\underline{a}_s^0 - \underline{1}\right)$$

is bounded for $\underline{y} \in \Theta$ if and only if

$$\sum_{i \in s} a_{si} x_i = 1 \qquad (2)$$

The Statistician interested in a minimax strategy will only consider a_{si}, $i \in s$ with (2). Therefore, the subset

$$T_0 = \{ \underline{y} \in T : \sum y_i = 0 \}$$

of Nature's pure strategies is of primary importance.

Let $s \in S$ be fixed and consider a mixed strategy π of Nature which is a discrete probability on Θ_0 giving rise to the pay-off

$$\sum \pi(\underline{y}) \left(\underline{a}_s^0 - \underline{1} \right)' \underline{y}\, \underline{y}' \left(\underline{a}_s^0 - \underline{1} \right)' = \left(\underline{a}_s^0 - \underline{1} \right)' V \left(\underline{a}_s^0 - \underline{1} \right)$$

where

$$V = \sum \pi(\underline{y})\, \underline{y}\, \underline{y}'$$

satisfies $V \underline{1} = \underline{0}$.

Subsequently, vectors and matrices are partitioned in accordance with common use. For a $N \times N$ matrix C, a N-vector $\underline{z}$ and $s \in S$ we write C_{ss} for the $n \times n$ submatrix composed of all c_{ij} with $i, j \in s$ and $\underline{z}_s$ for the n-vector consisting of z_i , $i \in s$.

Defining

$$\underline{a}_s(V) = \frac{V_{ss}^{-1} \underline{x}_s}{\underline{x}_s' V_{ss}^{-1} \underline{x}_s}$$

we have

$$(\underline{a}_s^0 - \underline{1})' V (\underline{a}_s^0 - \underline{1}) \geq (\underline{a}_s^0(V) - \underline{1})' V (\underline{a}_s^0(V) - \underline{1})$$

for all $\underline{a}_s$ with (2), i.e. $\underline{a}_s(V)$ is a best reply of the Statistician to V, as long as he is restricted to s (and (2)). This is an easy consequence from regression analysis. (See remark 1.)

Theorem 1 in combination with Lemma 5 show that a mixed strategy $\overset{*}{\pi}$ of Nature exists such that

$$\overset{*}{Q} = \sum \overset{*}{\pi}(\underline{y})\,\underline{y}\;\underline{y}'$$

has the following property:

$$\rho = \left(\underline{a}_s^0(\overset{*}{Q}) - \underline{1} \right)' \overset{*}{Q} \left(\underline{a}_s^0(\overset{*}{Q}) - \underline{1} \right)$$

does not depend on $s \in S$. Hence, all

$$\left(s, \underline{a}_s^0(\overset{*}{Q}) \right), s \in S$$

are best replies of the Statistician to Nature's mixed strategy $\overset{*}{\pi}$, defining $\overset{*}{Q}$.

A sampling strategy (p, t) is a mixed strategy of the Statistician, with pay-off $R(\underline{y}; p, t)$, $\underline{y}$ a pure strategy of Nature. In Theorem 2 we prove

$$\sup_{\underline{y} \in \Theta} R\left(\underline{y}; p, t\right) \geq \rho$$

for all strategies (p, t), i.e ρ is a lower bound for the maximal risk of sampling strategies.

Finally, we show in Theorem 3 that the equation

$$U = \frac{c^2}{\rho} \sum_{s \in S} p_s \,(\underline{a}_s^0(\overset{*}{Q}) - \underline{1})'\,\overset{*}{Q}(\underline{a}_s^0(\overset{*}{Q}) - \underline{1})$$

admits a solution $\overset{*}{p}_s$, $s \in S$ with $\sum \overset{*}{p}_s = 1$. For $\underline{x}$ and U close to 1/N and $I - \underline{1}\underline{1}'/N$, respectively, we have $\overset{*}{p}_s \geq 0$ for all $s \in S$, i.e. $\overset{*}{p} : s \to \overset{*}{p}_s$ is a design and, with

$$\overset{*}{t}(s, \underline{y}) = \Sigma\, a_{si}(\overset{*}{Q})\, y_i$$

it follows

$$R(\underline{y}; \overset{*}{p}, \overset{*}{t}) = \frac{\rho}{c^2}\, \underline{y}' U \underline{y}$$

and for all mixed strategies π of Nature and all mixed strategies (p, t) of the Statistician

$$\sum \pi(\underline{y})\ R(\underline{y}; \overset{*}{p}, \overset{*}{t}) \le \sum \overset{*}{\pi}(\underline{y})\, R(\underline{y}; \overset{*}{p}, \overset{*}{t}) = \rho$$

$$\le \sum \overset{*}{\pi}(\underline{y})\, R(\underline{y}; p, t)$$

i.e. $\overset{*}{\pi}$ and $(\overset{*}{p}, \overset{*}{t})$ form an equilibrium point of the game considered and ρ is the value of this game. As an immediate consequence,

$$\sup_{\underline{y}\in\Theta} R(\underline{y}; \overset{*}{p}, \overset{*}{t}) = \rho \le \sup_{\underline{y}\in\Theta} R\left(\underline{y}; p, t\right)$$

for all (p, t). Hence, $(\overset{*}{p}, \overset{*}{t})$ is minimax.

Remark 1. Consider the linear regression model

$$\underline{Y} = \underline{x}\,\beta + \underline{\varepsilon}$$

for $\underline{Y} = \left(Y_1, Y_2, \ldots Y_N\right)'$ where $\underline{\varepsilon}$ is a N-dimensional random vector with

$$E\,\underline{\varepsilon} = \underline{0}$$
$$\text{var}\,\underline{\varepsilon} = \sigma^2 V$$

Here, β and $\sigma > 0$ are (unknown) parameters. $V\,\underline{1} = \underline{0}$ implies

$$\sum_1^N Y_i = \beta$$

with probability 1; therefore, predicting $\sum_1^N Y_i$ coincides with estimating β. A linear predictor

$$\sum_{i\in s} a_{si} Y_i$$

is unbiased for $\sum_1^N Y_i$ if

$$E\left(\sum a_{si} Y_i - \sum_1^N Y_i\right) = 0$$

i.e. (2). Of all linear and unbiased predictors

$$\underline{Y}_s' \underline{a}_s(V) = \underline{Y}' \underline{a}_s^0(V)$$

has minimal variance:

$$E\left[\underline{Y}'\left(\underline{a}_s^0 - \underline{1}\right)\right]^2 = \left(\underline{a}_s^0 - \underline{1}\right)' V \left(\underline{a}_s^0 - \underline{1}\right)\sigma^2$$

$$\geq \left(\underline{a}_s^0(V) - \underline{1}\right)' V \left(\underline{a}_s^0(V) - \underline{1}\right)\sigma^2$$

$$= \frac{\sigma^2}{\underline{x}_s' V_{ss}^{-1} \underline{x}_s}$$

for all $\underline{a}_s \in A_s$,

$$A_s = \left\{ \underline{a} \in \Re^N : a_i = 0 \text{ for } i \notin s \ ; \ \underline{a}' \underline{x} = 1 \right\}.$$

Hence, $\underline{Y}' \underline{a}_s^0(V)$ is best linear unbiased (BLU) as an estimator of β and a predictor of $\sum Y_i$.

4. Preliminaries

In this section we derive results on eigen-vectors and –values of non-negative definite $N \times N$ matrices of the type

$$C + \underline{u}\,\underline{v}' .$$

With a few exceptions we will have $\underline{u} = \underline{v}$ in which case we use the notation

$$C + ?\,\underline{1}\,\underline{1}'?$$

with a diagonal matrix ? . Of special importance are matrices

$$C = D(I - \alpha \underline{1}\,\underline{1}')D$$

D always diagonal and often equal to I.

The vector $\underline{x}$ which is an eigen-vector of the matrix U defining the parameter space Θ will be essential in this section while other properties of U play no role.

We will have occasion to apply the following two lemmas.

Lemma 1: Assume C regular and $1 + v'C^{-1}u \neq 0$. Then

$$\left(C + \underline{u}\,\underline{v}' \right)^{-1} = C^{-1} - \frac{C^{-1}\underline{u}\,\underline{v}'C^{-1}}{1+\underline{v}'C^{-1}\underline{u}} \qquad (3)$$

and with $\underline{v} = \underline{u}$

$$(C + \underline{u}\,\underline{u}')^{-1}\underline{u} = \frac{C^{-1}\underline{u}}{1+\underline{u}'\,C^{-1}\underline{u}} \qquad (4)$$

Lemma 2: For $\alpha > 0$ and a diagonal $N \times N$ matrix Δ with $\underline{1}'?\,\underline{1} \neq 0$, consider

$$M = I - \alpha \underline{1}\,\underline{1}' + ?\,\underline{1}\,\underline{1}'?$$

with eigenvalues

$$\lambda_1 \geq \lambda_2 \geq K \geq \lambda_N$$

Then

$$\lambda_2 = \lambda_3 = \Lambda = \lambda_{N-1} = 1$$

Proof: For $\underline{u} \in \Re^N$

$$M\underline{u} = \lambda\underline{u}$$

is equivalent to

$$(1-\lambda)\,\underline{u} - \alpha\,\underline{1}\left(\underline{1}'\underline{u}\right) + ?\,\underline{1}\left(\underline{1}'?\,\underline{u}\right) = \underline{0} \tag{5}$$

Without restricting generality we assume linear independence of $\Delta\,\underline{1}$ and $\underline{1}$. Then, the equations $\underline{1}'\underline{u} = 0, \underline{1}'?\,\underline{u} = 0$ define a (N-2)-dimensional subspace with N-2 eigenvalues, all equal to 1.

Define $\mu = 1-\lambda$ and multiply (5) from the left by $\underline{1}'$ and $\underline{1}'?$, respectively, to obtain

$$\mu(\underline{1}'\underline{u}) - \alpha(\underline{1}'\underline{1})\,(\underline{1}'\underline{u}) + (\underline{1}'?\,\underline{1})\,(\underline{1}'?\,\underline{u}) = 0$$

$$\alpha(\underline{1}'?\,\underline{u}) - \alpha(\underline{1}'?\,\underline{1})\,(\underline{1}'\underline{u}) + (\underline{1}'?^{\,2}\underline{1})\,(\underline{1}'?\,\underline{u}) = 0$$

or equivalently

$$\begin{pmatrix} \mu - N\alpha & \underline{1}'?\,\underline{1} \\ -\alpha\underline{1}'?\,\underline{1} & \mu + \underline{1}'?^{\,2}\underline{1} \end{pmatrix} \begin{pmatrix} \underline{1}'u \\ \underline{1}'?\,\underline{u} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Assuming $\underline{1}'\underline{u} \neq 0$ or $\underline{1}'?\,\underline{u} \neq 0$ we derive

$$(\mu - N\alpha)\,(\mu + \underline{1}'?^{\,2}\underline{1}) + \alpha(\underline{1}'?\,\underline{1})^2 = 0$$

with solutions

$$\mu_1, \mu_2 = \frac{\alpha N - \underline{1}'?^{\,2}\underline{1} \pm \sqrt{\left(\alpha N - \underline{1}'?^{\,2}\underline{1}\right)^2 + 4\alpha\left[N\underline{1}'?^{\,2}\underline{1} - \left(\underline{1}'?\,\underline{1}\right)^2\right]}}{2}$$

satisfying

$$\mu_1\mu_2 = -\alpha\left[N\underline{1}'?^{\,2}\underline{1} - \left(\underline{1}'?\,\underline{1}\right)^2\right] \leq 0$$

because of the Cauchy-Schwarz inequality

$$\left(\underline{1}'?\,\underline{1}\right)^2 \leq N\,\underline{1}'?^{\,2}\underline{1}$$

Hence we cannot have $\mu_1 > 0$ and $\mu_2 > 0$ at the same time; therefore

$$\lambda_1 \geq 1 \geq \lambda_N .$$

Next we want to determine a diagonal matrix D such that

$$Q = D^{-1}(I - \frac{\underline{1}\underline{1}'}{n})D^{-1} + \underline{x}\,\underline{x}'$$

is non-negative definite with rank N-1 and $Q\underline{1} = \underline{0}$. As shown in Lemma 4 this is possible for $\underline{x}$ satisfying the weak condition (7) given in Lemma 3. In Theorem 1 we will prove a fundamental property of Q.

Lemma 3: Consider $\underline{x} = (x_1, x_2, \ldots x_N)'$ with $x = 1$ and

$x_i > 0$ for $i = 1, 2, \ldots N$.

A solution z_0 of

$$|N - 2n|\, z = \sum_{i=1}^{N} \sqrt{z^2 - x_i} \tag{7}$$

exists if and only if

$$|N - 2n| \geq \sum \sqrt{1 - x_i / x_o} \tag{8}$$

where

$$x_0 = \max\{x_1, x_2, \ldots x_N\}$$

The solution z_0 is unique.

Proof: Define

$$f(z) = |N-2n|\, z$$
$$g(z) = \sum_1^N \sqrt{z^2 - x_i}$$

For $z \geq \sqrt{x_o}$

$$g'(z) = z \sum_1^N \frac{1}{\sqrt{z^2 - x_i}} \; > \; N \; > \; f'(z) = |N-2n|$$

$$g''(z) = \sum \frac{1}{\sqrt{z^2 - x_i}} - z^2 \sum \frac{1}{\left(z^2 - x_i\right)^{3/2}}$$

$$= \sum \left(z^2 - x_i\right)^{-3/2} \left(z^2 - x_i - z^2\right)$$

$$= -\sum x_i \left(z^2 - x_i\right)^{-3/2} < 0$$

Therefore,

$$f(z) = g(z)$$

admits at most one solution. A solution exists if and only if

$$f\left(\sqrt{x_o}\right) \geq g\left(\sqrt{x_o}\right)$$

which is equivalent to

$$|N-2n| \sqrt{x_o} \geq \sum \sqrt{x_o - x_i}$$

i.e.

$$|N-2n| \geq \sum \sqrt{1 - x_i / x_o}$$

Lemma 4: Define $\kappa = \text{sgn}\ (N-2n)$ and

$$d_i = \frac{z_o + \kappa\sqrt{z_o^2 - x_i}}{x_i} \quad \text{for } i = 1, 2, \ldots N$$

$$D = \text{diag}(d_1, d_2, \ldots d_N)$$

Then

$$Q = D^{-1}(I - \frac{\underline{1}\,\underline{1}'}{n})D^{-1} + \underline{x}\,\underline{x}'$$

is of rank N-1 and non-negative definite with

$$Q\,\underline{1} = \underline{0}\ . \tag{9}$$

Proof: (9) is equivalent to

$$\frac{1}{d_i^2} - \frac{1}{n}\frac{1}{d_i}\sum\frac{1}{d_j} + x_i = 0 \quad ; \; i = 1, 2, \ldots N\,. \tag{10}$$

With

$$z = \frac{1}{2n}\sum\frac{1}{d_j} \tag{11}$$

(10) may be written

$$\left(\frac{1}{d_i} - z\right)^2 = z^2 - x_i \;\; ; \; i = 1, 2, \ldots N. \tag{12}$$

Now, z, $d_1, \ldots d_N$ solve (11), (12) if and only if for some $\varepsilon_1, \ldots \varepsilon_N = 1, -1$

$$\frac{1}{d_i} = z + \varepsilon_i \sqrt{z^2 - x_i} \quad ; \; i = 1, 2, \ldots N$$

and

$$z = \frac{1}{2n}\left(N\,z + \sum \varepsilon_i \sqrt{z^2 - x_i}\right)$$

i.e.

$$(N-2n)\ z = -\sum \varepsilon_i \sqrt{z^2 - x_i}$$

Define $\varepsilon_i = -\kappa$ for i=1, ... N. Then, the equations to solve are

$$\frac{1}{d_i} = z - \kappa\sqrt{z^2 - x_i} \ ; \ i = 1, \dots N$$

$$|N-2n|\ z = \sum \sqrt{z^2 - x_i} \quad .$$

Hence, a special solution is $z_o, d_1, \dots d_N, z_o$ with

$$|N-2n| z_o = -\sum \sqrt{z_o^2 - x_i}$$

and , for i = 1, 2, … N, d_i with

$$\frac{1}{d_i} = z_o - \kappa \sqrt{z_o^2 - x_i}$$

which is equivalent to

$$d_i = \frac{z_o + \kappa \sqrt{z_o^2 - x_i}}{x_i} \quad .$$

Further

$$Q = D^{-1}(I - \frac{\underline{1}'\underline{1}}{n} + D\underline{x}\,\underline{x}'D)D^{-1} = D^{-1}MD^{-1} \text{ , say.}$$

Since the assumptions of Lemma 2 are satisfied for M and 0 is an eigenvalue of M, all other eigen-values must be positive (in fact ≥ 1) and the rank and definiteness statements follow.

Theorem 1: For all $s \in S$,

$$\underline{a}_s(Q) = \frac{D_s \underline{1}_s}{\underline{x}_s' D_s \underline{1}_s} \tag{13}$$

$$\left(\underline{a}_s^0(Q) - \underline{1}\right)' Q \left(\underline{a}_s^0(Q) - \underline{1}\right) = 1 \tag{14}$$

Proof: Obviously

$$Q_{ss} = D_s^{-1}\left(I_s - \frac{\underline{1}_s \underline{1}_s'}{n}\right) D_s^{-1} + \underline{x}_s \underline{x}_s'$$

$$= \left(D_s^{-2} + \underline{x}_s \underline{x}_s'\right) - \frac{D_s^{-1} \underline{1}_s \underline{1}_s' D_s^{-1}}{n}$$

Hence, by (3)

$$Q_{ss}^{-1} = \left(D_s^{-2} + \underline{x}_s \underline{x}_s'\right)^{-1}$$

$$+ \frac{\left(D_s^{-2} + \underline{x}_s \underline{x}_s'\right)^{-1} \frac{D_s^{-1} \underline{1}_s \underline{1}_s' D_s^{-1}}{n} \left(D_s^{-2} + \underline{x}_s \underline{x}_s'\right)^{-1}}{1 - \frac{1}{n} \underline{1}_s' D_s^{-1} \left(D_s^{-2} + \underline{x}_s \underline{x}_s'\right)^{-1} D_s^{-1} 1_s}$$

Again by (3)

$$\left(D_s^{-2} + \underline{x}_s \underline{x}_s'\right)^{-1} = D_s^2 - \frac{D_s^2 \underline{x}_s \underline{x}_s' D_s^2}{1 + \underline{x}_s' D_s^2 \underline{x}_s}$$

and by (4)

$$\left(D_s^{-2} + \underline{x}_s \underline{x}_s'\right)^{-1} \underline{x}_s = \frac{D_s^2 \underline{x}_s}{1 + \underline{x}_s' D_s^2 \underline{x}_s} \quad .$$

Therefore

$$Q_{ss}^{-1}\underline{x}_s = \frac{D_s^2 \underline{x}_s}{1+\underline{x}_s' D_s^2 \underline{x}_s} + \frac{\left(D_s \underline{1}_s - \frac{D_s^2 \underline{x}_s \underline{x}_s' D_s \underline{1}_s}{1+\underline{x}_s' D_s^2 \underline{x}_s} \right) \frac{\underline{1}_s' D_s \underline{x}_s}{1+\underline{x}_s' D_s^2 \underline{x}_s} \frac{1}{n}}{1-\frac{1}{n}\left(n - \frac{\underline{1}_s' D_s \underline{x}_s \underline{x}_s' D_s \underline{1}}{1+\underline{x}_s' D_s^2 \underline{x}_s} \right)}$$

$$= \frac{D_s^2 \underline{x}_s}{1+\underline{x}_s' D_s^2 \underline{x}_s} + \left(D_s \underline{1}_s - D_s^2 \underline{x}_s \frac{\underline{x}_s' D_s \underline{1}_s}{1+\underline{x}_s' D_s^2 \underline{x}_s} \right) \frac{1}{\underline{1}_s' D_s \underline{x}_s}$$

$$= \frac{D_s \underline{1}_s}{\underline{x}_s' D_s \underline{1}_s} \quad .$$

Consequently,

$$\underline{x}_s' Q_{ss}^{-1} \underline{x}_s = 1$$

and

$$\underline{a}_s(Q) = \frac{Q_{ss}^{-1} \underline{x}_s}{\underline{x}_s' Q_{ss}^{-1} \underline{x}_s} = \frac{D_s \underline{1}_s}{\underline{x}_s' D_s \underline{1}_s}$$

which is (13) . (14) follows from (10) and

$$\left[\underline{a}_s^0(Q) - \underline{1}\right]' Q \left[\underline{a}_s^0(Q) - \underline{1}\right] = \left[\underline{a}_s^0(Q)\right]' Q \, \underline{a}_s^0(Q)$$

$$= \underline{a}_s'(Q) \, Q_{ss} \, \underline{a}_s(Q)$$

$$= \underline{x}_s' Q_{ss}^{-1} Q_{ss} \, Q_{ss}^{-1} \underline{x}_s$$

$$= 1 \; .$$

Remark 2. Let $\sigma^2 Q$ be the variance matrix of the residuals in a linear regression model. Then, the variance of the BLU-predictor for $\sum Y_i$, based on a sample s , is

$$\frac{\sigma^2}{\underline{x}_s' Q_{ss}^{-1} \underline{x}_s} = \sigma^2$$

and does not depend on s.

5. A lower bound for the maximal risk

Now, we are prepared to derive a lower bound of the maximal risk with respect to Θ. Note that here only weak assumptions concerning $\underline{x}$ are needed and that the matrix U defining the parameter space Θ has to satisfy $U\underline{x} = \underline{0}$, but otherwise is arbitrary. First we show in Lemma 5 that $\frac{c^2}{\text{tr}(QU)} Q$ can be expressed as a mixed strategy of Nature.

Lemma 5: Let U be non-negative definite of rank N-1 with (see (1))

$$U\underline{x} = \underline{0}$$

(see (1)). Then, a $N \times (N-1)$ matrix Z and positive probabilities

$$\pi_1, \pi_2, \ldots \pi_{N-1}$$

exist with

$$\underline{1}' Z = \underline{0} \tag{15}$$

$$Z' U Z = I_{N-1} \tag{16}$$

$$Q = \text{tr}(QU) \cdot Z \,\text{diag}(\pi_1, \ldots \pi_{N-1}) Z' \tag{17}$$

where Q is defined by (9) in Lemma 4.

Proof: Define

$$A = \left(I - \frac{1}{N}\underline{1}\underline{1}'\right) U \left(I - \frac{1}{N}\underline{1}\underline{1}'\right)$$

with eigenvalue decomposition

$$A = T?T'$$

where

$$T'T = I_{N-1}$$
$$T'\underline{1} = \underline{0}$$

$$\Delta = \operatorname{diag}\left(\delta_1, K\ \delta_{N-1}\right)$$

with $\delta_1, K\ \delta_{N-1} > 0$. Define further

$$B = T?^{\frac{1}{2}}T'$$
$$B^{+} = T?^{-\frac{1}{2}}T'$$

and consider the eigenvalue decomposition

$$BQB = C?\,C'$$

where

$$C'C = I_{N-1}$$
$$C'\underline{1} = \underline{0}$$
$$? = \operatorname{diag}\left(\lambda_1, K\ \lambda_{N-1}\right)$$

with $\lambda_1, K\ \lambda_{N-1} > 0$.

Defining $Z = B^{+}C$ equation (15) is obvious. Further, by $TT' = I - \frac{1}{N}\underline{1}\underline{1}'$,

$$Q = B^{+}C?\,C^{'}B^{+}$$
$$= Z?\,Z^{'}\ .$$

According to (15) we have

$$Z^{'}\,U\,Z = Z^{'}A\,Z$$
$$= C^{'}\,T?^{\ -\frac{1}{2}}\,T^{'}\,T?\,T^{'}\,T?^{\ -\frac{1}{2}}\,T^{'}C$$

giving (16). In addition

$$\mathrm{tr}(QU) = \mathrm{tr}(Z?\,Z^{'}U) = \mathrm{tr}(?\,Z^{'}UZ)$$

$$= \mathrm{tr}(?\,) = \sum \lambda_j$$

and with

$$\pi_i = \frac{\lambda_i}{\sum \lambda_j}\,;\ i = 1, 2, K\ \ N-1$$

we derive

$$Q = Z?\,Z^{'} = (\sum \lambda_j)\,Z\,\mathrm{diag}\left(\pi_1, K\ \ \pi_{N-1}\right)Z^{'}$$

$$= \mathrm{tr}\left(QU\right)\,Z\,\mathrm{diag}\left(\pi_1, K\ \ \pi_{N-1}\right)Z^{'}$$

i.e. (17).

Theorem 2: Consider $\underline{x} = \left(x_1,\ x_2, K\ \ x_N\right)^{'}$ with $x_i > 0$ for $i = 1, 2, K\ \ N$, $\sum x_i = 1$ and (see Lemma 3)

$$\left|\,N - 2n\,\right| \geq \sum \sqrt{1 - x_i / x_o}$$

where

$$x_o = \max\{x_1, \ldots x_N\}.$$

Let U be a $N \times N$ matrix of rank $N - 1$ with

$$U\underline{x} = \underline{0}$$

and define (see sections 1 and 3)

$$T = \{\underline{y} \in \Re^N : \underline{y}' \, U \, \underline{y} \le c^2\}.$$

$$\rho = \frac{c^2}{tr(QU)} \quad .$$

Then for all strategies (p, t)

$$\sup_{y \in T} R(\underline{y}; p, t) \ge \rho \; .$$

Proof: Define

$$\overset{*}{Q} = \rho \, Q \quad . \tag{18}$$

Then

$$\underline{a}_s\left(\overset{*}{Q}\right) = \underline{a}_s(Q)$$

and by (14)

$$\left[\underline{a}_s^0\left(\overset{*}{Q}\right) - \underline{1}\right]' \overset{*}{Q} \left[\underline{a}_s^0\left(\overset{*}{Q}\right) - \underline{1}\right] = \rho \tag{19}$$

for all $s \in S$. Now, consider a design p and an estimator t defined by $\underline{a}_s^0 \in A_s$. Obviously,

$$\sum p_s \left[\underline{a}_s^0 - \underline{1}\right]' \overset{*}{Q} \left[\underline{a}_s^0 - \underline{1}\right] \ge \sum p_s \left[\underline{a}_s^0\left(\overset{*}{Q}\right) - \underline{1}\right]' \overset{*}{Q} \left[\underline{a}_s^0\left(\overset{*}{Q}\right) - \underline{1}\right] = \rho$$

For $\underline{\pi} = (\pi_1, \ldots \pi_{N-1})'$ and $Z = (Z_1, \ldots Z_{N-1})$ defined in Lemma 5 we have

$$\overset{*}{Z}_i = c\, Z_i \in T$$

$$\overset{*}{Q} = \rho\, Q = \overset{*}{Z} \operatorname{diag}(\underline{\pi}) \overset{*}{Z}'$$

$$= \sum \pi_i \overset{*}{Z}_i \overset{*}{Z}_i'$$

Therefore

$$\rho = \sum p_s \sum \pi_i \left\{ \overset{*}{Z}_i' \left[\underline{a}_s^0 \left(\overset{*}{Q} \right) - \underline{1} \right] \right\}^2$$

$$\leq \sum p_s \sum \pi_i \left\{ \overset{*}{Z}_i' \left[\underline{a}_s^0 - \underline{1} \right] \right\}^2$$

$$\leq \max_i \sum p_s \left\{ \overset{*}{Z}_i' \left[\underline{a}_s^0 - \underline{1} \right] \right\}^2$$

$$= \max_i R\left(\overset{*}{Z}_i' ; p, t \right)$$

$$\leq \max_{\underline{y} \in T} R(\underline{y}; p, t) \quad .$$

6. Minimax strategies

We will show that a minimax strategy is obtained if the estimator $\overset{*}{t}$ introduced in section 5 is combined with an appropriate design $\overset{*}{p}$.

Lemma 6. Consider b_{ij} ; i, j = 1, 2, … N with

$$\sum_j b_{ij} = n\, b_{ii} \quad \text{for } i = 1, 2, \ldots N$$

and define for $s \in S$

$$b_s = \frac{1}{\binom{N-4}{n-2}} \sum_{\substack{i,j \in s \\ i<j}} b_{ij} - \frac{n-2}{\binom{N-2}{n-1}} \sum_{i \in s} b_{ii} + \frac{\binom{n-1}{2}}{\binom{N-2}{n}} \frac{\sum_{i=1}^{N} b_{ii}}{n} \tag{20}$$

Then, for $i, j = 1, 2, \ldots N$

$$b_{ij} = \sum_{s:\, i,j \in s} b_s$$

For the proof of Lemma 6 we refer to Chaudhuri (1971) and Gabler and Schweigkoffer (1990).

Theorem 3. Let $\underline{x}$ and U satisfy the conditions of Theorem 2. Define $D = \operatorname{diag}(d_1, \ldots d_N)$ and Q according to section 3 and $\overset{*}{t}$ according to section 5. Define further

$$m_i = \sum_j \frac{2u_{ij} - u_{ii} - u_{jj}}{d_j} \quad \text{for } i = 1, 2, \ldots N \tag{21}$$

$$k = \frac{\sum_i \dfrac{m_i}{2n - d_i \sum_j \frac{1}{d_j}}}{1 - \sum_i \dfrac{1}{2n - d_i \sum_j \frac{1}{d_j}}} \tag{22}$$

$$b_{ii} = \frac{k + m_i}{d_i (2n - d_i \sum \frac{1}{d_j})} \quad \text{for all } i = 1, 2, \ldots N \tag{23}$$

$$b_{ij} = \frac{d_i^2 b_{ii} + d_j^2 b_{jj} + 2u_{ij} - u_{ii} - u_{jj}}{2 d_i d_j} \quad \text{for } 1 \le i < j \le N \tag{24}$$

and b_s by (20). Then, the function $\overset{*}{p} : s \to \overset{*}{p}_s$ on S, defined by

$$\overset{*}{p}_s = \left(\sum_{j \in s} d_j x_j \right)^2 b_s / \operatorname{tr}(QU)$$

satisfies

$$\sum_s \overset{*}{p}_s = 1. \tag{25}$$

$\overset{*}{p}$ is a design and $(\overset{*}{p}, \overset{*}{t})$ is minimax with

$$\sup_{\underline{y} \in \Theta} R(\underline{y}; \overset{*}{p}, \overset{*}{t}) = \rho$$

provided $\underline{x}$ and U are close to $\underline{1}/N$ and $I - \frac{1}{N}\underline{1}\underline{1}'$, respectively.

Proof: From (22) and (23) we derive

$$\sum d_i b_{ii} = \sum_i \frac{k+m_i}{2n - d_i \sum \frac{1}{d_j}} = k \quad .$$

Hence, for i = 1, 2, … N and the fact that (24) remains true for i=j

$$\sum_j b_{ij} = \sum_j \frac{d_i^2 b_{ii} + d_j^2 b_{jj} + 2u_{ij} - u_{ii} - u_{jj}}{2 d_i d_j}$$

$$= \frac{1}{2}\left[d_i b_{ii} \sum_j \frac{1}{d_j} + \frac{1}{d_i} \sum_j d_j b_{jj} + \frac{1}{d_i} \sum_j \frac{2u_{ij} - u_{ii} - u_{jj}}{d_j} \right]$$

$$= \frac{1}{2}\left[d_i b_{ii} \sum_j \frac{1}{d_j} + \frac{k + m_i}{d_i} \right] \quad .$$

From (23) we obtain, therefore,

$$\sum_j b_{ij} = \frac{1}{2}\left[d_i b_{ii} \sum_j \frac{1}{d_j} + b_{ii}(2n - d_i \sum_j \frac{1}{d_j}) \right] \tag{26}$$

$$= n\, b_{ii}$$

Now

$$R\left(\underline{y};\overset{*}{p},\overset{*}{t}\right)=\sum_s \overset{*}{p}_s\left[\overset{*}{t}(s,\underline{y})-y\right]^2$$

$$=\sum_s\left(\sum_{j\in s} d_j x_j\right)^2 b_s\left[\sum_{i\in s}\frac{d_i}{\sum_{j\in s} d_j x_j}\cdot y_i-y\right]^2 \Big/ \operatorname{tr}(QU)$$

$$=\sum_s b_s\left[\sum_{i\in s} d_i\,(y_i-x_i y)\right]^2 \Big/ \operatorname{tr}(QU)$$

$$=\sum_{i,j} b_{ij} d_i d_j (y_i-x_i y)(y_j-x_j y) \Big/ \operatorname{tr}(QU) \qquad \text{by (26) and Lemma 6}$$

$$=\sum_{i,j} u_{ij} y_i y_j \Big/ \operatorname{tr}(QU) \qquad \text{by (23) and (24)}$$

$$=\underline{y}'U\underline{y} \Big/ \operatorname{tr}(QU) \qquad (27)$$

With π defined in Lemma 5 and $\overset{*}{Q}$ defined by (18) we derive from (27)

$$\sum \pi(\underline{y})\; R(\underline{y};\overset{*}{p},\overset{*}{t}) = \sum \pi(\underline{y})\;\; \underline{y}'U\underline{y}/\operatorname{tr}(QU)$$

$$=\operatorname{tr}\left(\sum \pi(\underline{y})\; \underline{y}\;\underline{y}'\cdot U\right)/\operatorname{tr}(QU)$$

$$=\operatorname{tr}(\overset{*}{Q}U)/\operatorname{tr}(QU)$$

$$=\rho \qquad \text{according to (18)}$$

On the other hand

$$\sum \pi(\underline{y})\; R(\underline{y};\overset{*}{p};\overset{*}{t}) = \sum \overset{*}{p}_s\,(\underline{a}_s^o(\overset{*}{Q})-1)'\;\overset{*}{Q}\,(\underline{a}_s^o(\overset{*}{Q})-1) = \rho\sum \overset{*}{p}_s$$

by (19), and (25) is proved.

Obviously, for $s \in S$, $\overset{*}{p}_s$ is a continuous function of U and $\underline{x}$ with limit $1\Big/\binom{N}{n}$ for $\underline{x}\to\underline{1}/N$ and $U\to I-\underline{1}\,\underline{1}'/N$. Hence, if $\underline{x}$ and U are close to $\underline{1}/N$ and $I-\underline{1}\,\underline{1}'/N$, respectively, we have

$$\overset{*}{p}_s \geq 0 \text{ for all } s \in S$$

and $\overset{*}{p}$ is a design.

Then, by (27)

$$\sup_{\underline{y} \in \Theta} R(\underline{y}; \overset{*}{p}, \overset{*}{t}) = \rho$$

and the minimaxity of $(\overset{*}{p}, \overset{*}{t})$ is a consequence of Theorem 2.

7. Concluding Remarks

The minimax strategy $(\overset{*}{p}, \overset{*}{t})$ derived in section 6 is independent of c . Subsequently, we consider two consequences of this independence.

It is common practice to characterize the performance of a strategy (p,t) by the mean squared error $R(\underline{y}; p, t)$ defined in section 1. However, there may be reasons to believe that $\underline{y}$ is close to

$$L = \{\lambda \underline{x} : \lambda \in \Re\}$$

where x_i is the size, measured appropriately, of unit i and $\underline{x} = (x_1, x_2, \ldots x_N)'$. Then, we will look for a strategy giving rise to a mean square error which is small for $\underline{y}$ close to L and 0 if $\underline{y} \in L$. This objective in mind we should base the selection of a strategy (p,t) on the risk function

$$\tilde{R} = \frac{R(\underline{y}; p, t)}{\underline{y}' U \underline{y}}$$

where U with $U \underline{x} = \underline{0}$ is non-negative definite and $\underline{y}' U \underline{y}$ is interpreted as squared distance of $\underline{y}$ from L . Now, $\tilde{R}$ is bounded on $\Re^N$ if and only if R is bounded on

$$\tilde{\Theta} = \{\underline{y} \in \Re^N : \underline{y}' U \underline{y} \leq 1\}$$

and

$$\sup_{\underline{y}\in\Re^N} \tilde{R}(\underline{y};p,t) = \sup_{\underline{y}\in\tilde{\Theta}} R(\underline{y};p,t)$$

Hence, the strategy $(\overset{*}{p},\overset{*}{t})$ is also minimax for $\tilde{R}$ and the parameter space $\Re^N$.

To derive the second consequence consider the following modification of the game described in section 3:

> *Nature selects c > 0 and subsequently* $\underline{y}\in\Theta=\{\underline{y}:\underline{y}'U\underline{y}\le c^2\}$. *The Statistician, without knowledge of c and* $\underline{y}$ *, selects* $s\hat{I}S$ *and* $\underline{a}\in A_s$. *Then, he has to pay* $[\underline{y}'(\underline{a}-\underline{1})]^2$.

Then,

$$\sum \pi(\underline{y})R(\underline{y};\overset{*}{p},\overset{*}{t}) \le \sum \overset{*}{\pi}(\underline{y})R(\underline{y};\overset{*}{p},\overset{*}{t}) \le \rho$$

$$\le \sum \overset{*}{\pi}(\underline{y})R(\underline{y};p,t)$$

for all discrete probabilities π on Θ and all strategies (p,t) , as earlier.

Note, however, that now ρ depends on Nature's strategy and is unbounded such that $(\overset{*}{p},\overset{*}{t})$ is no longer minimax in the sense defined. Under the present conditions the statistician may be interested in estimating c^2 , a problem which should not be easy to solve within the general setting of this paper.

Finally, we mention that the results presented may also be of interest for regression theory. A statistician adopting the strategy $(\overset{*}{p},\overset{*}{t})$ behaves as if he was analysing a linear regression model with variance of residuals in some neighbourhood of $\sigma^2\overset{*}{Q}$. He applies a mixture of best replies to $\sigma^2\overset{*}{Q}$ with weights protecting against certain deviations from $\sigma^2\overset{*}{Q}$, i.e. he behaves optimally with respect to $\sigma^2\overset{*}{Q}$ and, at the same time, takes a lower risk for models in the neighbourhood. See Stenger(1998) for details.

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