# Housing and Portfolio Choice: A Life Cycle Simulation Model

Max Flötotto

January 2006

The following text was written in the summer of 2005 as my diploma thesis. I thank Axel Börsch-Supan and Alexander Ludwig for superb advising and support. The Matlab code is available upon request – maxf@stanford.edu.

# Contents

# List of Tables

# List of Figures

# Chapter 1

# Introduction

Economists tend to see the world through the lens of their models. In attempting to find causal relationships in the complexities of economic interactions they need to abstract from many details. They make assumptions whenever this allows to simplify the exposition without affecting the results. Crucial assumptions, however, do affect the essential findings and should thus draw particular attention. The neglect of housing as a consumption and investment good in many models of portfolio choice theory may be such a crucial assumption. While this assumption my be justifiable in some cases, I will argue that housing – as a primary residence and not as an investment in a real estate fund – can play a central role in determining optimal asset demand.

First, housing is a prominent asset in most households' portfolios and responsible for a significant part of the households' expenses. Yao and Zhang (2005) report from the 2001 Survey of Consumer Finances that two thirds of U.S. households own their primary residence. For those, home value accounts for 55 percent of financial assets. Stock investments, however, account for less than 40 percent of the portfolio among those who do hold stocks at all. When looking at all households the number even shrinks to a meager twelve percent. Flavin and Yamashita (2002) find that on average the primary residence exceeds households' net worth until about age 50.

Second, housing investments are unusual in that they build on two motives – the consumption and the investment motive. The right to live in a particular residence can be interpreted as a consumption good that households can acquire either by renting from the owner or by buying their own housing stock from which housing services can then be derived. At the same time, households may follow an investment incentive in that they consider real estate as an attractive long-term investment that is not subject to inflation risk. The coexistence of two possibly conflicting motives can yield interesting results.

Third, housing investments are special in at least two ways, their illiquidity and their role as collateral for credit. While housing is an investment good with uncertain capital value not unlike other assets it cannot be traded in small shares and at low cost. Transaction costs in the housing market arise for a range of reasons. Brokers fees and taxes result in monetary costs that are more or less proportional to the property's value and according to Cocco (2005) in the order of eight to fifteen percent in the United States. Moreover, there are information and relocation costs which may vary substantially between house trades depending on characteristics such as the distance to the new house, preferences, etc. Those can come along as monetary costs and in the form of disutility. Transaction costs in the housing market act as an incentive to minimize house trades, in other words, the occurrence of adjustments of the housing stock. The second interesting peculiarity is the potential to use the housing stock as a collateral for credit. Housing investments are often highly leveraged in that investors are only required to invest a certain fraction of the house price up front, the so-called down payment.

Fourth, classic portfolio theory generally abstracts from housing as an additional illiquid asset class with its role as credit collateral. In the classic models elegant results can be derived both in the static and dynamic framework. The predictions' empirical validity though is weak. I do not propose housing as a satisfying solution to portfolio choice theory's many puzzles. However, I will show that the integration of housing into a model can have a significant effect on households' optimal portfolio demand over the life cycle. Young households invest strongly into housing as they attempt to reduce the number of future house trades. Often credit constrained they show very low financial assets other than housing. If asset market participation involves costs, an assumption that is often made in life cycle simulation models, the reduction of savings due to the presence of housing can make asset market participation unattractive for significant shares of the young (Cocco (2005)).

In order to illustrate the impact of housing on portfolio demand I construct a partial equilibrium model of portfolio demand over the life cycle. The model incorporates housing as an additional good in its dual role as nondurable consumption and investment good. The key features of the housing good are its illiquidity – house sales involve a significant transaction cost – and its collateral role, the households' credit line depends on the current value of the house. Households face three sources of risk: uncertain labor income, risky stock returns, and a risky house price. As I cannot find closed form solutions of the dynamic optimization problem I solve the model numerically in order to find optimal policies rules, i.e. the optimal behavior of the household given the state of the world. Using those, it is straightforward to simulate the model.

The remainder of this thesis is organized as follows. Chapter 2 discusses classic models of portfolio choice theory both in a static and dynamic framework. In each case I will present the effect of integrating housing into the analysis. Chapter 3 provides a brief overview of the empirical evidence on households' portfolios and also mentions ways in which the theoretical literature attempts to solve the remaining puzzles. Chapter 4 conveys the presentation of my simulation model and the calibration part. The computational strategy is explored in Chapter 5. Results of the simulation are discussed in Chapter 6 which also entails a discussion of possible extensions. Chapter 7 concludes.

# Chapter 2

# Housing and Portfolio Choice Theory

Before presenting an overview of households' portfolio allocation decisions in Chapter 3 I first concentrate on basic theoretical predictions of the portfolio choice literature. To be more exact, I augment a number of canonical portfolio choice models with a stylized housing good. Incorporating housing hardly changes the results in some cases. Other models, however, become significantly more complicated by the introduction of housing as can be seen further below.

In many models housing plays a dual role; it is both, a durable consumption good and an asset class of its own. In this section I first look at static portfolio choice models that integrate housing simply as an additional risky asset not unlike stocks. I then introduce illiquidity into the analysis which is an important feature of housing for most investors. Secondly, I will show that standard results carry over to dynamic and life cycle settings even in the presence of housing as an investment good. Third, in many models housing has an intrinsic value for the household – utility is derived from housing services that are related to the stock of housing owned. Adjusting the stock is costly and households will therefore only infrequently optimize their housing stock. In between those adjustments standard results of portfolio choice theory can be shown to hold.

## 2.1 Static Models with Housing

**Housing as an Additional Liquid Asset.** At this point I deliberately simplify housing's role in investors' portfolios to its investment dimension. Although most households do not exceed a handful of house trades in the course of their lifetime I concentrate on a simple model where housing is a liquid asset with uncertain capital value not unlike other assets. In this framework one might think of housing as a real estate investment fund; its share price is uncertain and investors can buy or sell shares of more or less any amount. Not surprisingly, the investor's optimization problem does not change much. Housing simply provides an additional investment opportunity.

In the very basic one period portfolio selection framework which is originally due to Harry Markowitz (1952)[1] investors care only about the mean and variance of their portfolio. In order to incorporate housing into this model I simply add the risky housing investment to $n$ other risky and a risk-free asset with known return $R^f$. Denote the vector of expected excess returns as $\boldsymbol{\mu}$ and the variance-covariance matrix of returns as $\boldsymbol{\Omega}$,

$$\boldsymbol{\mu} = \begin{pmatrix}\mu_1 & \ldots & \mu_n & \mu_h\end{pmatrix}', \qquad \boldsymbol{\Omega} = \begin{pmatrix}\boldsymbol{\Sigma} & \boldsymbol{\Gamma}_{s\,h} \\ \boldsymbol{\Gamma}'_{s\,h} & \sigma_h^2\end{pmatrix}$$

where $\boldsymbol{\Sigma}$ stands for the usual covariance matrix of asset return other than housing, $\sigma_h^2$ for the variance of the housing return, and $\boldsymbol{\Gamma}'_{s\,h} = [\sigma_{1\,h}, \ldots, \sigma_{n\,h}]$ for the vector of covariances between the individual asset returns and the return on the housing investment. In Markowitz' framework a portfolio is efficient if the vector of optimal portfolio shares $\boldsymbol{\alpha}$ minimizes the portfolio variance given a fixed expected return $\bar{R}$.

$$\min_{\boldsymbol{\alpha}} \boldsymbol{\alpha}'\boldsymbol{\Omega}\,\boldsymbol{\alpha} \quad \text{subject to} \tag{2.1}$$
$$\boldsymbol{\alpha}'\boldsymbol{\mu} + R^f = \bar{R}.$$

The result is very standard: the risky asset's optimal portfolio share depends on its expected excess return, the conditional variance-covariance matrix of returns, and the Lagrange multiplier with its

[1]Or more recently Markowitz (1992).

known interpretation as the coefficient of relative risk aversion as in Samuelson (1970)[2].

$$\boldsymbol{\alpha} = \begin{pmatrix}\alpha_1 & \dots & \alpha_n & \alpha_h\end{pmatrix}' = \boldsymbol{\Omega}^{-1}\frac{\boldsymbol{\mu}}{\theta} \tag{2.2}$$

Individual preferences enter the expression for optimal portfolio shares only through the coefficient $\theta$. Thus, Tobin's (1958) mutual fund separation theorem applies in this simple case – portfolios differ in scale, i.e. in the proportion of funds invested in the efficient combination of risky assets, but not in the composition of the risky portfolio. Depending on the covariances between housing and equity returns the introduction of housing as an additional asset is likely to affect the efficient Markowitz portfolio.[3] However, nothing essentially new is added to the analysis by the introduction of the liquid housing good.

**Housing as an Illiquid Asset.** As argued in the introduction adjusting one's housing stock can be costly for a variety of reasons. Therefore, I now introduce the first key feature of housing investments, their illiquidity. Taking this assumption to the extreme I model housing investments as irreversible. The owned housing stock is then simply a constraint on the investor's portfolio selection problem.

The following exposition is still set in the static mean-variance framework where housing enters only as an investment good without any intrinsic value.[4] Now, however, the risky housing investment is exogenously fixed for the period of interest to $\bar{\alpha}_h$. The investor's problem then becomes

$$\begin{aligned} \min_{\boldsymbol{\alpha}} \boldsymbol{\alpha}'\boldsymbol{\Omega}\,\boldsymbol{\alpha} \quad & \text{subject to} \\ & \boldsymbol{\alpha}'\boldsymbol{\mu} + R^f = \bar{R} \\ & \alpha_h = \bar{\alpha}_h. \end{aligned} \tag{2.3}$$

The solution is straightforward and delegated to the appendix. The optimal portfolio share for each risky asset other than housing is given

[2] This relies on CRRA specification of the wealth dependent utility function.

[3] Housing is not redundant and thus held in equilibrium as long as the other assets do not span it on the $\mu$-$\sigma$-plane.

[4] This part builds strongly on Pelizzon and Weber (2003).

by

$$\alpha_i = \mathbf{\Sigma}^{-1}\frac{\mu_i}{\theta} - \alpha_h \mathbf{\Sigma}^{-1}\mathbf{\Gamma}_{sh}. \tag{2.4}$$

In the presence of an illiquid housing investment the optimal portfolio shares consist of two parts. First, the investor optimally chooses an efficient Markowitz portfolio just as in the unconstrained problem in equation (2.2) which is identical to the standard case without housing altogether. Second, financial investments are used to hedge the house price risk, the risk inherent in the return to housing. That second part depends crucially on the covariances between housing and asset returns $\mathbf{\Gamma}_{sh}$. Thus, an illiquid housing investment has a non-trivial effect on the optimal financial investment portfolio. This effect depends not only on the covariance of asset and housing returns, but also on the given level of the housing stock relative to total wealth, the 'housing constraint', a term coined by Flavin and Yamashita (2002). If that level depends on other factors, such as age or wealth, interesting patterns might emerge.[5]

Flavin and Yamashita consider portfolio selection problems in a mean-variance framework at different points over the life cycle and assume an exogenous pattern for the "housing constraint". In their model younger households who show large housing to net worth ratios are forced into highly leveraged positions. Hence, a young household has an incentive to reduce her portfolio's risk by paying down a part of the mortgage or by investing in safe assets instead of stocks. Older households with lower housing to net worth ratios optimally hold larger parts of their wealth in risky assets. Hence, even without preference heterogeneity and constant investment opportunities the optimal portfolio of financial assets varies over the life cycle. Note, however, that this finding depends crucially on the rigid assumption of an exogenous housing to net worth ratio. An endogenous determination of the demand for housing is absent from the model.

[5]The obvious idea that wealthier households are likely to prefer larger, more expensive houses does not suffice to motivate exogenous variation in the housing constraint. It is the ratio of housing to total wealth that determines the constraint.

## 2.2 A Standard Life Cycle Model with Housing

Merton's (1969) basic result of optimal portfolio allocation in a static problem has been extended to multi period settings. Merton himself has proved the equivalence of dynamic and one period portfolio choice problems under the assumption that the investor's wealth dependent utility function is either logarithmic or of CRRA form, whereby the latter also requires investment opportunities to be constant over time. In such an environment it follows that myopic portfolio choice is optimal.

In this section I will demonstrate that Merton's result carries over to the life cycle framework. In order to do this, I present a standard life cycle model that features housing. However, I restrict my attention to a very simple setup where housing is only an additional investment opportunity in the household's optimization problem. It does not yet enter the utility function directly and it is no longer illiquid. After presenting this very stylized model I explore the effects of human wealth on our benchmark result – the optimal portfolio shares' independence from household characteristics except for the degree of risk aversion.

**Life Cycle Models.** In the traditional life cycle model that builds upon work from Modigliani and Brumberg (1954), Ando and Modigliani (1963), and Friedman (1957) households save in order to smooth consumption over the course of their lives which is optimal under the standard assumption of concave utility. Young households with relatively low earnings borrow against future income and later begin to pay back their debt and accumulate savings as income grows. After retirement the wealth is spent on consumption in excess of public pension benefits and capital income. The simple framework has been extended in many directions. In order to highlight the main point I abstract from complications such as the bequest motive, more realistic borrowing constraints, and time inconsistent preferences at this point. Neither will I consider household heterogeneity in terms of financial sophistication, varying attitudes toward risk or impatience.

**Setup of the Model.** In this very simplified model which originally builds upon Samuelson (1969) the household lives for $T$ periods. As there is no bequest motive the household will consume all her remaining wealth in period $T$. For simplicity she does not receive any labor income for now, but enters the model endowed with $W_0$. Her initial wealth can be used in two ways: immediate consumption or saving and investment in a risk-free asset, stocks or housing. The household's investments generate an overall return that is a weighted average of the individual returns.

$$R^p = R^f + \alpha_s(R^s - R^f) + \alpha_h(R^h - R^f)\,. \tag{2.5}$$

In the absence of labor income today's savings have to finance the present discounted value of future consumption. The household's intertemporal budget constraint is thus given by

$$W_t - C_t = \sum_{j=t+1}^{T} \prod_{s=t+i}^{j} (1 + R_s^p)^{-1} C_j \tag{2.6}$$

which can be transformed into the dynamic budget constraint[6]

$$W_{t+1} = (W_t - C_t)(1 + R_{t+1}^p)\,. \tag{2.7}$$

I will assume the often used specification of utility with a constant coefficient of risk aversion $\theta$. As stated above, there is no bequest motive and housing does not enter the utility function in this model.

$$u(C_t) = \frac{C_t^{1-\theta} - 1}{1-\theta} \tag{2.8}$$

[6]Pushing (2.6) one period forward and rearranging leads to:

$$\begin{aligned} W_{t+1} &= C_{t+1} + \sum_{j=t+2}^{T} \prod_{s=t+2}^{j} (1 + R_s^p)^{-1} C_j \\ &= (1 + R_{t+1}^p) \sum_{j=t+1}^{T} \prod_{s=t+1}^{j} (1 + R_s^p)^{-1} C_j \end{aligned}$$

which can be combined with the intertemporal budget constraint to find the dynamic budget constraint (2.7).

**The Household Problem.** The household chooses a consumption and an investment plan for each period to maximize expected discounted lifetime utility subject to the budget constraint.

$$\max_{\{C_t\},\{\alpha_{st}\},\{\alpha_{ht}\}} \quad U_0 = \sum_{t=1}^{T} \beta^t u(C_t) \tag{2.9}$$
$$\text{subject to (2.7)}$$

The problem can be formulated recursively. The Bellman equation with control variables $\{C, \alpha_s, \alpha_h\}$ and state $\{W\}$ then becomes

$$\mathcal{V}_t(W_t) = u(C_t) + \beta \mathbb{E}\big[\mathcal{V}_{t+1}(W_{t+1})\big] \,. \tag{2.10}$$

In the last period the household consumes all her remaining wealth as there is no bequest motive such that the value function $\mathcal{V}_T(W_T)$ equals $u(W_T)$.

$$\mathcal{V}_T(W_T) = u\big((W_{T-1} - C_{T-1})(1 + R^p_T)\big) \tag{2.11}$$

In period $T-1$ the Bellman equation takes on the following form that can be modified slightly as $(W_{T-1} - C_{T-1})^{1-\theta}$ is known with certainty at time $T-1$. The maximum is taken with respect to $C_{T-1}$, $\alpha_{st}$, and $\alpha_{ht}$.

$$\mathcal{V}_{T-1}(W_{T-1}) =$$
$$= \max \left\{ u(C_{T-1}) + \beta \mathbb{E}\left[ \frac{1}{1-\theta} (W_{T-1} - C_{T-1})^{1-\theta} (1 + R^p_T)^{1-\theta} \right] \right\} \tag{2.12}$$
$$= \max \left\{ u(C_{T-1}) + \beta \frac{1}{1-\theta} (W_{T-1} - C_{T-1})^{1-\theta} \, \mathbb{E}\left[ (1 + R^p_T)^{1-\theta} \right] \right\} \tag{2.13}$$

At this point it becomes clear that the maximization problem can be separated into two distinct ones. The household has to find both, an optimal consumption and an optimal investment plan. The allocation of savings among different assets, however, is unaffected by the total level of wealth. The optimal asset shares $\alpha_s$ and $\alpha_h$ are set to maximize the last part of the value function $\mathbb{E}\left[(1 + R^p_T)^{1-\theta}\right]$. Wealth only enters as a constant multiplicative factor into the maximization.

**Merton's Result.** Given the CRRA form for the utility function and assuming jointly lognormally distributed returns for equity and housing investments optimal portfolio shares depend only on the first and second moments of the return processes. This is again the famous result that is originally due to Merton (1969) and illustrated nicely in Campbell and Viceira (2001). Using small letters for logarithms and matrix notation with $\boldsymbol{\alpha} = \binom{\alpha_s}{\alpha_h}$, $\boldsymbol{r} = \binom{r_s}{r_h}$, $\boldsymbol{\sigma}^2 = \begin{pmatrix} \sigma_s^2 \\ \sigma_h^2 \end{pmatrix}$, $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_s^2 & \sigma_{sh} \\ \sigma_{sh} & \sigma_h^2 \end{pmatrix}$, and $\boldsymbol{\iota} = \binom{1}{1}$, Merton's result can be stated neatly as

$$\boldsymbol{\alpha}_t^{opt} = \frac{1}{\theta}\, \boldsymbol{\Sigma}_t^{-1}(E_t \boldsymbol{r}_{t+1} - r_{t+1}^f \boldsymbol{\iota} + \frac{\boldsymbol{\sigma}_t^2}{2})\,. \tag{2.14}$$

The derivation can be found in the appendix. Note that if we shut off the return covariances we get back to the standard result of the better known case with only one risky asset. With $\sigma_{sh} = 0$ the solution for $\alpha_s^{opt}$ becomes

$$\alpha_s^{opt} \quad = \quad \frac{E(r^s - r^f) + \frac{\sigma_s^2}{2}}{\theta \sigma_s^2}\,. \tag{2.15}$$

## 2.3 Separation of Investment and Consumption Motive

Flavin and Yamashita (2002) see the household's demand for housing as being 'overdetermined'. Consumers' housing needs can differ from an optimal holdings of real estate from the investor's point of view. In theory, rental markets for housing services could separate the decision on housing investment and consumption of housing services. In fact not only the poorest households who cannot afford the initial down payment rent their dwelling. However, rental housing is not a perfect substitute for owner-occupied housing in many settings.

The classic reference for this problem comes from Henderson and Ioannides (1983) who identify an externality from renting. In their model the amount of housing services that the household derives from her apartment depends on the size or quality of the housing stock, the so-called housing capacity, and the rate of utilization. Due to problems of asymmetric information and incomplete contracts the landlord cannot assess the true utilization of the property and thus recover the true cost of maintenance. However, from the landlord's point of view real estate is not different from any other asset. Thus, in asset market equilibrium the additional cost of higher utilization is eventually paid for by the tenant in terms of a higher rent. Henderson and Ioannides show that in equilibrium utilization rates are higher for renters than for owner-occupiers as the former cannot credibly commit to low utilization rates, i.e. to treat the property well. Hence, owning dominates renting under certainty. If returns are stochastic the uncertain future wealth implied by gains or losses in house value can partially or fully offset this advantage. However, Henderson and Ioannides show that even under uncertainty people in a certain income range distort their investment and consumption decision and owner-occupy rather than rent.

Fernandez-Villaverde and Krueger (2002) add another twist to that argument. Housing's role as a collateral for credit makes owning property even more attractive as it is a means to insure against future income uncertainty. On the other hand, housing investments cannot generally be fully refinanced with a mortgage. The necessary down payment could force especially younger households to reduce nondurable consumption significantly in order to save for the upfront cost of buying a house. The attractiveness of owning is further reduced by the higher transaction costs – it is certainly reasonable to assume higher relocation costs for home-owners when compared to renters.

Last but not least there may be institutional reasons for preferring owning to renting. The most obvious is the differential treatment in the tax system. Depreciation allowances for landlords and the deductibility of mortgage interest payments make owning very attractive.[7]

As illustrated above there are good reasons for a dominance of owning over renting which cannot, however, account for the widespread occurrence of renting even among the wealthy. In the remainder of this thesis I will abstract from the existence of rental markets and concentrate on the effects of home purchases, though. The main reason being the desire to keep the model tractable. An explicit treatment of the decision whether to own or rent can be found in Platania and Schlagenhauf (2000).

[7] Cf. Flavin and Yamashita (2002).

## 2.4 Housing Services in the Utility Function

In the models presented so far housing was just another risky investment opportunity. I did allow the housing stock to be illiquid, but so far there was no intrinsic value attached to holding housing stock. There is, of course, a strong incentive to invest in housing. Consider a tenant who pays a monthly rent. The pure fact that she spends a considerable part of her earnings on the rent, no matter whether it is a fifth or a third, demonstrates that households derive utility from housing services. In the following section I therefore consider utility functions that depend fully or in part on the consumption of durable goods such as housing services.

**Durable Consumption.** In their seminal contribution Grossman and Laroque (1990) introduce a utility function that depends exclusively on a durable good that generates consumption services. That durable good also appears as a component of the household's total wealth. Despite the fact that their approach is more general and applicable to all durable consumption goods, such as e.g. automobiles, I will henceforth call that good housing for the purpose of relating the model to this thesis.

In this model houses come in various sizes but are indivisible once bought. The owner derives utility from the house she lives in – there is no rental market to separate the consumption from the investment decision. Adjusting the housing stock in excess of what is lost due to depreciation is only possible by selling the total stock and buying a new house of the wanted size. However, selling one's property leads to a proportionate transaction cost an assumption that builds on agency problems in the real estate market, broker fees, and taxes. In the absence of this friction the consumer would keep his durable good consumption at a constant proportion to her wealth. That obviously cannot be optimal in the presence of transaction costs. Here, small variations in wealth will not lead to an adjustment of the housing stock.

Grossman and Laroque solve this dynamic model as an optimal stopping problem where the housing stock's illiquidity enters as a fixed cost. They find that the consumer optimally sets a target level for the ratio of liquid wealth to the housing stock where liquid wealth refers to total wealth less the cost of selling the current house. She only acts when that ratio reaches an upper or lower threshold. Hence, the home purchase decision is endogenous and fully rational. Numerical simulations show that even a small transaction cost of about five percent of the purchase price lead to very infrequent house trades about every twenty to thirty years. The household no longer equates the marginal utility of consumption with the marginal utility of wealth at all points of time. Optimal consumption is no longer a smooth function of wealth and the Consumption based CAPM fails. However, the investor still holds mean-variance efficient portfolio at all times and the standard CAPM continues to hold.

**Nondurable and Durable Consumption.** Flavin and Nakagawa (2004) as well as Pelizzon and Weber (2003) provide extensions to Grossman and Laroque's path breaking model in that they allow for the coexistence of durable and nondurable consumption in the households' utility function. That allows them to study the impact of housing on the dynamics of nondurable consumption and the explicit modeling of house price risk. The two articles differ most and for all in that Flavin and Nakagawa consider a special case as they assume a block diagonal variance-covariance matrix, i.e. zero covariances between the return to housing and the returns to financial assets.

The models are set in continuous time and at each instant the household has to decide whether it is optimal to adjust the housing stock immediately. That decision depends on the value of the program conditional on selling the house versus not selling the house. If the current stock of housing is far off its optimum and endogenous or voluntary house trade occurs. In addition, one could imagine exogenous or forced house trades which could be caused by events such as e.g. death, retirement, change in marital status, or the appearance of children. Imaging that the house is not sold at time $t$ equal to zero. By continuity, there must be an interval $(0, s)$ during which housing stock adjustments can be ignored. For that interval, the current housing stock becomes a state variable and optimal policy rules for nondurable consumption and financial investments can be derived. Optimal nondurable consumption is related to the standard equality of marginal utility and the derivative of the value function with respect to wealth. The optimal investments in risky assets can be shown to follow[8]

$$\alpha_i = \underbrace{\left( \frac{-\frac{\partial \mathcal{V}}{\partial W}}{\frac{\partial^2 \mathcal{V}}{\partial W^2} W} \right)}_{\tilde{\theta}} \mathbf{\Sigma}^{-1} \mu_i - \alpha_h \mathbf{\Gamma}_{ih} \mathbf{\Sigma}^{-1} \,. \tag{2.16}$$

As emphasized by Pelizzon and Weber this expression is analogous to the solution of the static mean-variance analysis in (2.4). Note the effect of the introduction of housing as a second state variable on $\tilde{\theta}$, a term that signifies relative risk aversion. In the simpler case above, this term used to be a constant – the coefficient of relative risk aversion – that depended solely on the curvature parameter of the instantaneous utility function. Now, however, the term is no longer necessarily constant even with CRRA utility. It depends on the curvature of the value function and thus on the state variables. The property that risk aversion varies with the state is also considered by Grossman and Laroque who find that the household is relatively more risk averse immediately after purchasing a new home. Housing wealth thus affects portfolio allocation through its impact on investors' relative risk aversion.

The second term of (2.16) signifies once again a hedge portfolio. Hence, in the presence of the illiquid housing stock in its dual form as a durable consumption good and as an asset households choose the efficient Markowitz portfolio conditional on their risk attitude. However, risky financial assets are also used to hedge the exposition to the constrained asset housing. This last motive is independent of the investor's risk attitude. In other words, if stock adjustments are costly and therefore infrequent the optimal portfolios in between house trades will be affected by house price risk. The hedge term disappears in Flavin and Nakagawa's solution as they assumed a block diagonal covariance matrix, i.e. zero covariance between the house price and all asset prices. Hence, in their model the portfolio choice problem can be separated between real and financial assets and the traditional CAPM holds in between house trades.

[8]The derivation of (2.16) is sketched in the appendix.

# Chapter 3

# Household Portfolios

Having presented some findings of the theoretical literature on portfolio choice in the preceding section it is now time to confront the predictions with empirical evidence. In the following paragraphs I will first point to the important separation of the investor's participation and portfolio composition decision. In a second step I will highlight some prominent patterns in households' portfolio allocations.

Theoretical models generally simplify the broad range of available assets to a handful of choices, a risk free and one or more risky assets. When looking for testable patterns in the data an important preliminary decision needs to be made on the definition of asset classes. In particular it is critical to define the term risky assets. Direct stockholdings and indirect stockholdings through mutual funds and retirements accounts must clearly be classified as risky assets. Foreign assets are also risky as they are subject to interest and exchange rate risk. While long-term government bonds are often seen as almost risk-free, they are also subject to interest rate risk and inflation risk given that they are generally not indexed to the price level. Finally, equity in privately owned businesses and real estate can also be included by the definition. In what follows I use the classification employed by Bertaut and Starr-McCluer (2002) and present numbers for three sets of assets: direct and indirect stockholding, risky financial assets, and total risky assets. The second definition adds corporate, foreign and mortgage backed bonds to direct and indirect stockholdings. The third definition also includes business equity and investment real estate. All numbers are taken from the Survey of Consumer Finances.

The famous rise of the equity culture – a dramatic increase in the proportion of stockholders in many countries during the 1990s – really builds on two effects. It is certainly true that stocks have come into the focus of many investors for the first time during the last decade and a half. People who used to invest exclusively in safe assets began to add stocks or more importantly mutual funds to their portfolios. On the other hand, those who already used to hold stocks shifted larger parts of their portfolios into that direction. For instance, Guiso and Japelli (2002) find that the two effects had about equal impact on total investments in risky assets in Italy.

Hence, when looking at households' investment decisions two distinct concepts need to be understood. Before allocating the available funds to different assets the household has to decide whether to invest in a certain asset class at all – the participation decision. In a second step the household determines individual portfolio shares conditional on the participation decision. That is the portfolio composition. In order to disentangle the underlying mechanisms one has to look at the two question separately.

## 3.1 Determinants of Portfolio Decisions

The empirical literature has identified a handful of characteristics that seem to have first-order effects on investors' participation and allocation decisions. The most important among those are the investor's wealth, age, and educational attainment. For each one I will quickly describe prominent theoretical arguments that might explain the underlying effect before presenting the empirical findings.

**Wealth.** Standard portfolio choice theory predicts that richer households will hold more risky portfolios if risk aversion depends negatively on wealth. To be more exact, richer households hold more risky assets if absolute risk aversion decreases in wealth and they invest a larger share of their portfolio in risky assets if relative risk aversion decreases in wealth.[1] In the benchmark case with constant relative risk aver-

[1] The very common Constant Relative Risk Aversion formulation of instantaneous utility implies decreasing absolute risk aversion. Cf. Gollier (2002).

sion richer households are simply scaled up versions of their poorer counterparts. If this widely used assumption is made wealth predicts higher absolute investments into risky assets.

Barsky, Juster, Kimball, and Shapiro (1997) use survey data and experimental designs to estimate individual coefficients of risk aversion. They go on and try to relate those to the respondents' risk-taking behavior in the real world. They find that lower risk aversion does indeed increase the likelihood of holding stocks.[2] If there is exogenous ex-ante variation in risk attitude only the less risk averse will engage strongly in risky high-return activities and – given that they are lucky enough – become rich. Hence, when looking at the data ex post one should expect to find a correlation of risk-taking and wealth.

However, Carroll (2002b) argues that the empirical evidence fails to detect a link between wealth growth and expressed ex-ante risk aversion. He concludes that ex-ante variation in risk attitude cannot explain the observed correlation between wealth and risk taking. He proposes a variation of the standard model in which wealth enters the utility function directly in Carroll (2000). The utility gain from wealth can result from very different motives such as the desire for philanthropic bequests or pure greed. Carroll models wealth as a luxury good and builds upon Max Weber's (Weber 1958) idea that love of wealth for its own sake is the spirit of capitalism. His model predicts realistic wealth accumulation patterns and portfolio shares especially for the wealthy elderly.

Another argument builds upon capital market imperfections. Concentrating on wealth held in privately owned business Quadrini (1999) tries to find an explanation for the concentration of entrepreneurs among the very rich. He supposes a minimum efficient scale for private businesses and credit market imperfections that require entrepreneurial investments to be largely self-financed due to adverse selection and moral hazard problems. Taken together the two assumptions can indeed explain the low probability of households with low or moderate wealth to become engaged in entrepreneurial activities. However, the model cannot account for the higher risk in the financial portfolio of business owners when compared to other households.

[2] Note, however, that their estimated coefficients are smaller than classical portfolio theory suggests.

**Age.** The investor's age can affect the portfolio decision for a number of reasons. It is a striking common feature of most pieces of financial advice that young investors are told to select relatively risky portfolios in order to profit from the wealth-generating potential of the equity premium.[3] Older investors are told to rebalance and shift resources into safer assets as they near retirement. While the advice stands in contrast to the simple predictions of the classical theory presented earlier it can be rationalized by taking the investor's human capital or human wealth into account. The following presentation of this argument builds upon the presentation in Campbell and Viceira (2001).

So far I have concentrated on the household's financial wealth and abstracted from labor income. While this is valid only for institutional investors most households receive periodic income flows, transfers or pension benefits. If one considers the expected present discounted value of future income as an asset – call it human wealth – current income flows can be interpreted as dividend payments on the implicit holdings of that asset. In contrast to most financial assets human wealth is not easily tradable. In the absence of slavery a person who has sold claims against her future labor income cannot be forced to continue working. Hence, moral hazard problems render human wealth nontradable.

The household can anticipate future income streams and calculate her human wealth $HW$ as the present discounted value of future labor income. The model in section 2.2 can be extended by human wealth by noting that total wealth $W$ is simply the sum of financial wealth $\hat{W}$ and human wealth $HW$. Equation (2.14) still determines the optimal portfolio shares such that the optimal holdings of asset $i$ amount to $\alpha_i(\hat{W} + HW)$. Portfolio shares, however, are generally expressed as the part of financial wealth that is accorded to a particular asset, $\hat{\alpha}$. The optimal shares for the financial portfolio can be calculated as

$$\hat{\alpha}_i = \frac{\alpha_i(\hat{W} + HW)}{\hat{W}} = \alpha_i\left(1 + \frac{HW}{\hat{W}}\right) > \alpha_i\,. \tag{3.1}$$

In order to reach the optimal allocation of total wealth the explicit asset holdings are adjusted to compensate for implicit holdings of human wealth. As a result, the optimal financial portfolio is tilted to-

[3] Cf. Canner, Mankiw, and Weil (1997).

ward risky assets in the presence of risk-free labor income. This effect is stronger the larger the ratio of human wealth to financial wealth. Intuitively, the investor who already holds most of her total wealth in safe assets – by her implicit holdings of human wealth – optimally shifts a larger part of her financial wealth into risky assets. Optimal portfolio shares are thus tilted toward stocks for investors who hold risk-free, non-tradable human wealth.

The ratio of human wealth to financial wealth changes over the life cycle. An employed young investor still expects many years of future labor income and typically disposes of little financial wealth. Human wealth might even increase during the first few years of the work life; the investor comes closer to her high income years such that the high earnings are discounted for less years. However, human wealth peaks early in life and then falls and reaches zero when the investor dies.[4] Financial wealth in the denominator should be increasing with age until retirement. The ratio of human wealth to financial wealth in equation (3.1) thus falls over the life cycle and only rebounces only late in life.

Hence, a young investor should optimally accord a larger fraction of her financial wealth to risky assets than an older investors even if the level of financial wealth and risk characteristics are identical. Note the this model leads to a prediction that is the exact opposite of the result in Flavin and Yamashita (2002). Particularly young household are expected to show the corner solution of $\hat{\alpha}$ equal to one – i.e. full exposure to the stock market – in the presence of borrowing constraints.

Future earnings cannot be predicted with certainty. However, it can be shown that all employed investors should optimally tilt their portfolios toward risky assets under the assumption of idiosyncratic labor income risk. The higher the variance of labor income the less pronounced this tilt becomes. If labor income is positively correlated with asset returns the effect reverses and the investor optimally reduces her exposure to risky assets compared to a situation with risk-free labor income. In that case the implicit holdings of human wealth bear more resemblance to a risky asset and are thus compensated by shifting

[4] One could argue that human wealth reaches zero at retirement if public pension benefits are seen as annuities that depend on earlier contributions to the public pension system or if the retirees rely entirely on private pensions.

funds to risk-free investments. A positive correlation of income and return shocks can be a plausible assumption given that many investors fail to diversify internationally or even among industries.[5]

Households can influence the value of their human wealth for example by varying their future labor supply and by investments in education and career choices. The endogenous determination of the labor supply is explored by Bodie, Merton, and Samuelson (1992) who find that the additional margin of adjustment provides the household with insurance against negative return shocks. That enables especially younger households to take more risks in their financial investments.

A different argument can be made regarding the attitude toward risk. In their study cite above Barsky, Juster, Kimball, and Shapiro also construct age patterns of risk aversion. They find a hump shape – low risk aversion, and thus high participation in the stock market, among the young and the elderly, and a high aversion to risk in the middle ages. According to Paxson (1990), however, those households at risk of facing binding liquidity constraints and uncertain needs – younger households – should invest more in safe and liquid assets. To the same result leads another argument. It is sensible to expect education, experience and especially financial sophistication to increase with age. For example, the simplifying assumption taken in King and Leape (1987) that information about stockholding arrives exogenously and randomly over time implies that the probability of stockholding increases in age. Hence, older investors are prone to show higher market participation rates and more diversification in their portfolios.

It will soon be clear that the empirical evidence stands in striking contrast to the first two arguments. Younger households do not hold higher conditional portfolio shares in risky assets and, more importantly, they participate in the stock market to a much lower degree.

**Education.** The impact of education on investment decisions can be understood in terms of informational requirements. It can be argued that the investment in risky assets requires the investor to gather more information on expected returns, the risk characteristics and eventual institutional peculiarities like tax treatments. If a higher level of eduction enables the household to fulfill the informational requirements

[5]Cf. discussion in chapter six of Campbell and Viceira (2001).

faster and at lower cost risky investments are relatively more attractive for the more educated. Put differently, poorly educated households might simply be unaware of the existence of a particular asset class.[6] Guiso and Japelli (2002) construct an index of financial information simply by counting the number of different assets known by the household head and find that the index is highly correlated with asset market participation.

## 3.2 Empirical findings

**Participation Rates.** One of the most persistent findings of the empirical literature is the fact that many households do not hold risky assets at all. For instance, Bertaut and Starr-McCluer (2002) report that – despite the upward trend in participation (Table 3.1) – for more than half of all U.S. households the only financial asset held is a savings or checking account.

The household's wealth has a very strong influence on the participation decision in that wealthier households show much higher participation rates. Comparing participation in risky financial assets for different quartiles of the wealth distribution Guiso, Haliassos, and Japelli (2002) report astonishing numbers as can be seen in Table 3.2.[7] The vast majority of households in the richest quartile hold risky financial assets. At the lower end of the wealth distribution less than five percent of the households hold stocks directly or indirectly.

A robust finding of many empirical studies is a hump-shaped age profile of participation in risky assets which can be found in Table 3.3. Participation is low among the young, rises with age, peaks in the 50-59 age bracket and declines during retirement.

**Portfolio composition.** With regard to portfolio shares conditional on participation empirical studies identify similar determinants as for the participation decision. Their effects are far less marked, though.

[6] King and Leape (1987) report that 40 percent of non-stockholders in the Survey of Consumer Financial Decisions stated as a reason that they did not know enough about the stock market.

[7] The first three panels report the proportion of investors by gross financial wealth quartiles. The last panel reports the proportion of investors by total asset quartiles.

**Table 3.1:** Proportion of households investing in risky assets

| | Year | | | |
|---|---|---|---|---|
| | 1983 | 1989 | 1995 | 1998 |
| Direct stockholding | 19.1 | 16.8 | 15.2 | 19.2 |
| Direct and indirect stockholding | n.a. | 31.6 | 40.4 | 48.9 |
| Risky financial assets | n.a. | 31.9 | 40.6 | 49.2 |
| Total risky assets | n.a. | 46.4 | 51.6 | 56.9 |

Source: Guiso, Haliassos, and Japelli (2002)

**Table 3.2:** Proportion of households investing in risky assets, by asset quartiles

| | Quartile | | | | Top |
|---|---|---|---|---|---|
| | I | II | III | IV | 5% |
| Direct stockholding | 1.4 | 6.9 | 20.6 | 47.9 | 70.1 |
| Direct and indirect stockholding | 4.4 | 38.3 | 66.0 | 86.7 | 93.7 |
| Risky financial assets | 4.4 | 38.6 | 66.4 | 87.2 | 93.8 |
| Total risky assets | 15.4 | 48.0 | 70.8 | 93.2 | 98.8 |

Source: Guiso, Haliassos, and Japelli (2002)

**Table 3.3:** Proportion of households investing in risky assets, by age

| | Age | | | | | |
|---|---|---|---|---|---|---|
| | 20-29 | 30-39 | 40-49 | 50-59 | 60-69 | 70+ |
| Direct stockholding | 11.8 | 16.0 | 21.2 | 24.8 | 23.7 | 18.2 |
| Dir. and ind. stockh. | 34.3 | 51.8 | 58.3 | 61.4 | 47.1 | 32.4 |
| Risky financial assets | 34.5 | 51.8 | 58.5 | 61.5 | 47.9 | 33.4 |
| Total risky assets | 38.7 | 58.6 | 67.0 | 68.4 | 59.2 | 42.4 |

Source: Guiso, Haliassos, and Japelli (2002)

Wealthier households hold a larger part of their portfolio in risky financial assets (Table 3.4). However, when controlling for other household characteristics the effect largely disappears and the residual can easily be accounted for with theories of decreasing relative risk aversion or economies of scale in portfolio management.

The age profile for conditional asset shares is one of the more controversial questions in the empirical literature. Many studies detect a hump-shape for the share of risky assets in the portfolio over the life cycle, a pattern that can also be found in Table 3.5. However, that hump is generally rather flat.

**Housing Stock and Debt.** Flavin and Yamashita (2002) report life cycle patterns of asset holdings for home-owners. Table 3.6 illustrates that the youngest home-owners under age 30 are highly leveraged with a housing value to net worth ratio of 3.5.[8] The ratio falls over the life cycle as the household accumulates wealth. A similarly clear pattern is evident for the ratio of mortgage to net wealth. Note that the figures point to the existence of a down payment requirement in the financing of home equity. One might even be tempted to find support for the notion of a credit constraint used in my model in Chapter 4: the constraint becomes more stringent the older the investor becomes. The total mortgage is well below the value of the house, the ratio of the two falling significantly with age.[9]

Debt can also appear as unsecured consumer credit. Despite the higher borrowing costs many households especially younger ones take on substantial amounts of consumer credit. Davis, Kübler, and Willen (2005) present evidence that 74 percent of U.S. households have at least one credit card with 44 percent of all households showing a positive balance after the most recent repayment in 1995.

[8] Mean asset-to-net worth ratio of home-owners in the 1989 PSID Wealth Data.

[9] This is only intended as a back-of-the-envelope calculation. The figures represent cohort means and cannot be aggregated that easily.

**Table 3.4:** Conditional asset shares, by asset quartiles

| | Quartile | | | | Top |
|---|---|---|---|---|---|
| | I | II | III | IV | 5% |
| Direct stockholding | 32.9 | 33.0 | 25.6 | 34.9 | 37.9 |
| Direct and indirect stockholding | 40.7 | 45.0 | 49.0 | 60.4 | 64.0 |
| Risky financial assets | 40.7 | 44.9 | 49.0 | 61.3 | 65.2 |
| Total risky assets | 29.5 | 24.8 | 23.4 | 59.2 | 69.4 |

Source: Guiso, Haliassos, and Japelli (2002)

**Table 3.5:** Conditional asset shares, by age

| | Age | | | | | |
|---|---|---|---|---|---|---|
| | 20-29 | 30-39 | 40-49 | 50-59 | 60-69 | 70+ |
| Direct stockholding | 22.5 | 28.3 | 29.4 | 32.7 | 37.5 | 41.3 |
| Dir. and ind. stockh. | 52.0 | 53.4 | 61.0 | 61.4 | 60.8 | 57.9 |
| Risky financial assets | 52.1 | 53.7 | 61.8 | 62.1 | 61.4 | 59.4 |
| Total risky assets | 44.4 | 43.0 | 52.9 | 58.8 | 56.2 | 56.1 |

Source: Guiso, Haliassos, and Japelli (2002)

**Table 3.6:** Ratio of house value and mortgage to net worth

| | Age | | | | | |
|---|---|---|---|---|---|---|
| | 18-30 | 31-40 | 41-50 | 51-60 | 61-70 | 71+ |
| House value | 3.511 | 2.366 | 1.588 | 0.969 | 0.757 | 0.648 |
| Mortgage | 2.833 | 1.671 | 0.882 | 0.319 | 0.171 | 0.038 |

Source: Flavin and Yamashita (2002)

## 3.3 Promising Extensions to Standard Theory

As illustrated above there are substantial differences in the patterns of participation rates and portfolio composition. The often cited factors wealth, age, and education exert their influence mainly at the preliminary stage where the household decides whether to invest in a risky asset class at all. However, classical theory concentrates on the optimal allocation of funds between safe and risky assets given that both are held. What needs to be explained, though, is the concave age profile of participation in risky assets as well as the positive correlation between participation and the two determinants wealth and education.

Haliassos and Bertaut (1995) show that expected-utility maximizers should always be willing to hold an arbitrarily small amount invested in the risky asset. Intuitively, an investor optimizes not only along the consumption - savings margin, but also along the portfolio margin. What matters for the risk and return trade-off is not the asset's idiosyncratic risk, but its covariance with the marginal utility of consumption and at zero stockholdings stocks have zero covariance with $u'(C_{t+1})$. In the following paragraphs I present a handful of promising extensions to the basic model that may help to reconcile the observed low participation rates among the young with classical portfolio choice theory.

**Participation Costs.** One promising way to reconcile theoretical models with the data lies in the introduction of participation costs. Those costs can come along as monetary transaction costs, minimum investment requirements, or information costs and what really matters are the perceived rather than the actual costs.

First, monetary transaction costs arise for stock investments and investments in mutual funds in the form of entry and liquidation, or brokerage fees.[10] Significant reductions generally apply only to very large investments. Note that these costs are variable and thus do not

[10] Haliassos and Bertaut (1995) cite Aiyagari and Gertler (1990) who report the following retail broker fees with discount brokers charging 30-70 percent less: minimum counselling charges of $30-50 and inversely proportional fees in the order of 2-8 percent. It is reasonable to assume a further decline in fees until today.

fit well to the fixed costs of equity market participation that are often used to achieve realistic participation rates among the young and the less wealthy. Second, there are minimum investment requirements which can here do a better job. Households with low financial wealth can well be deprived of the possibility to invest in equity if their desired stockholdings are smaller than the minimum investment requirement. Haliassos and Bertaut (1995), for instance, consider minimum investment requirements of $500 realistic. Third, the future investor may well experience disutility when looking for a broker, opening an account, and more importantly when searching for information concerning the potential gains and losses of a particular asset. It is reasonable to expect the informational costs to be relatively high for individual stock investments and a little less so for mutual funds.

Information costs are lower for the better educated, if education is a good proxy for the ability to collect and process information. An interesting point is made by Guiso and Japelli (2002): if participation costs only consisted of monetary costs and minimum investment requirements, households at the very top of the wealth distribution should hold efficient portfolios. However, that is not the case in most countries. While the proportion of households in the richest quartile holding stocks directly or indirectly is indeed very high in the U.S., the numbers are less promising for other countries with only 41.6 percent in Germany and 54.6 percent in Italy.[11] Information costs are not necessarily correlated with wealth and might therefore explain why so many rich households do not invest in risky assets.

**Liquidity Needs.** First, liquidity needs can lead to interesting effects when considered together with participation costs. In the case of entry costs – as with mutual fund investments – the annual expected cost is smaller the longer the investment horizon. Hence, investors with short-term liquidity needs are less likely to pay the participation cost. Liquidity needs are typically high for households with uncertain needs or for those who face binding liquidity constraints, which both applies particularly to young households. Moreover, the argument also applies to the young who intend to purchase a home and save for the

[11]Cf. Guiso, Haliassos, and Japelli (2002).

required down payment.[12]

Second, liquidity constraints can have effects on their own. In his seminal empirical study on the effect of liquidity constraints on consumption behavior Zeldes (1989) notes that a liquidity constraint need not bind in order to affect current behavior. If the constraint binds with some positive probability consumption will be lower for any risk-averse individual. The desire to hold positive amounts of nonhuman wealth has been coined precautionary savings.

**Interest Rate Wedge.** Davis, Kübler, and Willen (2005) explore the life cycle effects of an interest rate wedge, a borrowing rate that exceeds the risk free interest rate. They present evidence of a borrowing rate that exceeds the risk free return by about six to nine percentage points. In their life cycle simulation the demand for equity is minimized if the borrowing rate equals the expected return on equity. In that case no investor borrows to buy equity. A lower borrowing rate induces investors to hold a debt-financed portfolio. A higher borrowing rate on the other hand discourages borrowing for consumption-smoothing purposes which leads households to reach a positive financial position earlier and hold more equity in later ages.[13] Their simulation matches observed life-cycle patterns for both equity holdings and unsecured borrowing. However, it does not achieve realistic bond portfolios. As Haliassos and Bertaut (1995) note, households that are pushed to the corner solution of no stockholdings necessarily show zero holdings in the risk free asset, too.

Laibson, Repetto, and Tobacman (2001) study the surprising coexistence of cheap collateralized and expensive non-collateralized debt. They propose dynamically inconsistent time preferences as a solution to the finding that many households accumulate large holdings of illiquid assets while paying high interest on their revolving credit card debt.

[12] Cf. Guiso and Japelli (2002).

[13] A higher borrowing rate makes debt-financed equity positions unattractive. Hence, at younger ages a borrowing rate that exceeds the expected equity return can lead to lower equity holdings.

# Chapter 4

# The Model

## 4.1 Setup of the Model

**Preferences.** In my model, the household gains instantaneous utility from nondurable consumption and housing services. While the former need to be purchased anew in every period housing services depend on the real stock of housing that the household owns in the period of interest. The underlying relation is very simple: the household receives one unit of housing services for every unit of housing owned. The units could be interpreted as a function of the dwelling's properties such as size and quality. I use a standard CRRA utility function for the composite good and a Cobb-Douglas function to aggregate current consumption and housing services. The weight parameter $\phi$ determines preferences toward consumption and housing services. $\theta$ denotes the curvature parameter.[1] I abstract from a bequest motive in order to concentrate on the direct effect of housing. The household discounts the future using a constant factor $\beta \leq 1$.

$$u(C_t, H_t) = \frac{\left(C_t^{\phi} H_t^{1-\phi}\right)^{1-\theta}}{1-\theta} \tag{4.1}$$

[1] In the presence of illiquid housing as an additional state variable the coefficient of relative risk aversion is no longer identical to the curvature parameter $\theta$. In this model risk attitude is state-dependent and the coefficient of relative risk aversion is not necessarily constant. See Flavin and Nakagawa (2004) for an in-depth discussion. However, I still describe the utility function as CRRA as this term is generally used for a function of this particular form.

**Demographics.** As this model abstracts from overlapping generations aspects I do not need separate indices for cohorts and age. Hence, I set $t$ equal to one when the household enters the model. The household works until $\hat{T}$ which I assume to be exogenous and deterministic and leaves the model at age $T$. When solving the model for optimal behavior, however, I follow Hubbard, Skinner, and Zeldes (1994) and include age-specific conditional survival probabilities where $\pi_t$ stands for the probability that a household who reached age $t$ is still alive at age $t+1$. Those affect the age-specific discount factor that the household uses to aggregate current and future utility. Given the assumptions concerning the individual discount factor and the conditional survival probabilities the expected discounted lifetime utility becomes

$$U_0 = \mathbb{E}\Bigg[\sum_{t=1}^{T} \beta^t \Bigg(\prod_{j=1}^{t} \pi_j\Bigg) u(C_t, H_t)\Bigg] \tag{4.2}$$

**Labor income.** The household receives an exogenous stream of labor income $Y_{kt}$ in each period of her working life. That income stream depends on two factors, a deterministic wage profile that incorporates the effects of age, ability and education, $f(t, Z_{kt})$, and a stochastic component $u_{kt}$. In order to keep the computation feasible I abstract from the often used separation of the stochastic component in permanent and transitory shocks as well as aggregate and idiosyncratic shocks.[2] Instead, I model the stochastic part of labor income as an idiosyncratic lognormal i.i.d shock.

$$Y_{kt} = f(t, Z_{kt}) \exp\{u_{kt}\} \quad \text{for } t < \hat{T} \tag{4.3}$$

$$u_{kt} \sim N(0, \sigma_u^2) \tag{4.4}$$

During retirement, i.e. after age $\hat{T}$, the household receives a risk-free pension. Again for simplicity I assume the pension to be deterministic and independent of an individual's prior earning shocks.

$$Y_{tk} = \vartheta_k \quad \text{for } t > \hat{T} \tag{4.5}$$

[2]Campbell and Viceira (2001), for instance, model log income as $f(t, Z_{kt}) + v_{kt} + u_{kt}$ where the first part of the stochastic component, $v_{kt}$, follows an $AR(1)$ process.

**Housing Stock.** Housing plays a dual role in this model. It is both a durable consumption good and a vehicle for investment. The household receives housing services to an amount that is directly proportional to the size of the current housing stock.[3] In each period a fraction $\delta$ of the stock is lost due to wear and tear. Moreover, housing is an investment good with uncertain capital value $P$ which is a relative price that represents the price of one unit of housing in terms of consumption goods. Note that the house price $P$ does not affect the amount of housing services that a given stock of housing generates. The services depend exclusively on the stock's size. For computational reasons that I elaborate on in Chapter 5 the house price is introduced as a stochastic process with a deterministic exponential trend.

$$P_t = \exp\{gt + \eta_t\} \tag{4.6}$$

The return to housing is twofold. First, housing services yield instantaneous utility and can thus be seen as a dividend on the cumulated housing investment. Second, the housing price follows the process given in (4.6) from which the financial return on housing can be calculated as

$$\begin{aligned} R_t^h &= \log\{P_t\} - \log\{P_{t-1}\} \\ &= g + \eta_t - \eta_{t-1} \\ &= g + \Delta\eta_t \,. \end{aligned} \tag{4.7}$$

Housing in this model should not be seen as an investment in a diversified real estate fund that can be bought and sold at low cost. It rather refers to the household's primary residence and is thus traded as a single piece. Housing is severely illiquid. When adjusting the housing stock the household has to sell her old property, incurring a transaction cost that amounts to $\lambda$ times the old house's current value $PH_{-1}$. Denote the gross investment in the housing stock by $PI^H$. Hence, the quantity of housing owned is determined by

$$H_t = \begin{cases} (1-\delta)H_{t-1} & \text{for } I_t^H = 0 \\ (1-\lambda)(1-\delta)H_{t-1} + I_t^H & \text{for } I_t^H \neq 0\,. \end{cases} \tag{4.8}$$

[3] Quality improvements of the house appear simply as enlargements of the housing stock. Depreciation also affects solely the size of the stock.

**Financial Assets.** There are three financial instruments in this model – stocks, bonds and a mortgage. The household can invest nonnegative amounts in stocks which yield the stochastic return $R^s$ and the risk free asset, bonds, that has a certain return $R^f$. The sum of stocks and bonds determines the household's liquid assets that I will call $A$. The third instrument is a mortgage, $D$, which requires fixed interest payments $R^d - 1$ in every period. One can express the current holdings of each asset in relation to liquid assets $A$ by $\alpha_i$ for $i \in \{s, f, d\}$. I rule out short-sales in all three assets.

$$\alpha_s \geq 0 \tag{4.9}$$

$$\alpha_f \geq 0 \tag{4.10}$$

$$\alpha_d \geq 0 \tag{4.11}$$

The return on stocks is modeled as the sum of the risk-free return, the expected excess return $\mu = \mathbb{E}[R^s - R^f]$ and a stochastic component $\epsilon \sim N(0, \sigma_s^2)$. I allow $\epsilon$ to be correlated with the stochastic element of the housing return $\Delta\eta$ and denote the correlation by $\rho_{sh}$.[4]

$$R_t^s = R^f + \mu + \epsilon_t \tag{4.12}$$

The maximum mortgage that the household is allowed to take out depends on the value of the current housing stock. Real estate can thus be financed with a loan except for the down payment, the fraction $\kappa$ of the house's value that has to be paid upfront. Note that I link the mortgage to the property's value and not directly to the investment in the housing stock – the additional resources that the household receives by taking out a mortgage can be used for immediate consumption, financial investments or housing investments. The mortgage can be renegotiated at no cost in every period. This assumption is computationally attractive – the mortgage enters as a control variable, but not as an additional state. This significantly reduces the time needed for solving the model numerically. It is reasonable to expect tighter restrictions on borrowing by the elderly who face a higher

[4]Note that I hold financial investment opportunities constant over time. As illustrated in Section 2.2 this constitutes an important assumption for standard portfolio choice theory.

mortality risk. Hence, I indexed the down payment fraction by $t$ which allows me to vary the constraint over the life cycle.

$$D_t \leq (1 - \kappa_t) P_t H_t \tag{4.13}$$

**Minimum Investment Requirement.** The baseline model features an important additional constraint, the minimum investment requirement. In contrast to Cocco (2005) and many others I do not impose a fixed financial cost of asset market participation. In his framework, for instance, investors can access the stock market at no cost for the rest of their lives once they have paid the upfront participation cost. Hence, those costs can refer either to the monetary entry fees that mutual fund ask for or to the monetary equivalent of informational requirements, the time and effort of first-time investors who have to reach a certain level of financial sophistication.

The minimum investment requirement does not involve a monetary or utility cost for the investor. The constraint simply requires stock investments to have a minimum size $\Psi$. This feature can of course be institutional in the sense that brokers do not offer trades or depots below a threshold. Moreover, households might simply believe that it is impossible to buy shares in very low numbers or that the variable utility costs of gathering information exceed the possible gains from very small investments. As an additional computational benefit I avoid including an additional state variable by chosing the minimum investment requirement instead of the widely used fixed costs of equity market participation.

**Budget Constraint.** The household begins each period with a level of liquid wealth $LW$ that depends on last period's financial investments, the debt taken out, as well as the realization of the applicable return process. Remember that $A$ denotes liquid assets, the sum of stocks and bonds. Rewriting the expression for liquid wealth will be helpful below.

$$\begin{aligned} LW_{t+1} &= Stocks_t\, R^s_{t+1} + Bonds_t\, R^f_{t+1} - D_t R^d_{t+1} \\ &= A_t\big(R^f_{t+1} + \alpha_s (R^s_{t+1} - R^f_{t+1})\big) - D_t R^d_{t+1} \end{aligned} \tag{4.14}$$

I follow Deaton (1991) and express cash-on-hand $X$ as the sum of liquid wealth and current labor income.

$$X_{t+1} = LW_{t+1} + Y_{t+1} \tag{4.15}$$

The household can use her resources which consist of cash on hand and the debt taken out to purchase the nondurable consumption good, to invest in the financial assets stocks and bonds, or to adjust her housing stock.

$$X_t + D_t = A_t + C_t + P_t I_t^H \tag{4.16}$$

Combining (4.14), (4.15), and (4.16) leads to the household's dynamic budget constraint, i.e.

$$\begin{aligned} X_{t+1} = &\left(X_t + D_t - C_t - P_t I_t^H\right)\left(R_{t+1}^f + \alpha_s(R_{t+1}^s - R_{t+1}^f)\right) \\ &- D_t R_{t+1}^d + Y_{t+1} \, . \end{aligned} \tag{4.17}$$

## 4.2 The Household Problem

The household maximizes expected discounted lifetime utility subject to the dynamic budget constraint, the law of motion for the housing stock, the stochastic processes, and the constraints on the financial instruments which have all been introduced in the previous section.

$$\max_{\{C_t, D_t, \alpha_t^s, I_t^H\}} U_0 = \mathbb{E}\left[\sum_{t=1}^{T} \beta^t \left(\prod_{j=1}^{t} \pi_j\right) u(C_t, H_t)\right] \tag{4.18}$$

The problem has three state variables, cash-on-hand $X_t$, last period's housing stock, $H_{t-1}$, and the current house price $P_t$. Let $\mathcal{V}_t$ denote the household's period $t$ expectation of life-time welfare given that the state variables take on the values $X_t$, $H_{t-1}$, and $P_t$ and that the policy rules for the controls $C_t$, $D_t$, $\alpha_t^s$ and $I_t^H$ are optimal for the current and all future periods. The maximization problem can be restated recursively using Bellman's principle of optimality as

$$\begin{aligned} &\mathcal{V}_t(X_t, H_{t-1}, P_t) = \\ \max_{C_t, D_t, \alpha_t^s, I_t^H} &\left\{u(C_t, H_t) + \beta \pi_t \mathbb{E} \mathcal{V}_{t+1}(X_{t+1}, H_t, P_{t+1})\right\} \\ &\text{subject to} \end{aligned} \tag{4.19}$$

*laws of motion for the states*

$$X_{t+1} = A_t\big(R^f_{t+1} + \alpha_s(R^s_{t+1} - R^f_{t+1})\big) - D_t R^d_{t+1} + Y_{t+1} \tag{4.20}$$

$$H_t = \begin{cases} (1-\delta)H_{t-1} & \text{for } \; I^H_t = 0 \\ (1-\lambda)(1-\delta)H_{t-1} + I^H_t & \text{for } \; I^H_t \neq 0 \end{cases} \tag{4.21}$$

*the return, income and house price process*

$$Y_t = \begin{cases} f(t, Z_t)\exp\{u_t\} & \text{for } t \leq \hat{T} \\ \vartheta & \text{for } t > \hat{T} \end{cases} \tag{4.22}$$

$$P_t = \exp\{gt + \eta_t\} \tag{4.23}$$

$$R^s_t = R^f_t + \mu + \epsilon_t \tag{4.24}$$

*constraint 1: 'credit constraint'*

$$D_t \leq \begin{cases} (1-\kappa_t)(1-\delta)P_t H_{t-1} & \text{for } I^H_t = 0 \\ (1-\kappa_t)P_t\left((1-\lambda)(1-\delta)H_{t-1} + I^H_t\right) & \text{for } I^H_t \neq 0 \end{cases} \tag{4.25}$$

*constraint 2: 'nonnegative financial investments'*[5]

$$A_t = X_t + D_t - C_t - P_t I^H_t \geq 0 \tag{4.26}$$

*constraint 3: 'no short sales for stocks'*

$$\alpha^s_t \geq 0 \tag{4.27}$$

*constraint 4: 'no short sales for bills'*

$$\alpha^s_t \leq 1 \tag{4.28}$$

*constraint 5: 'minimum investment requirement'*

$$\alpha^s_t = \begin{cases} \alpha^s_t & \text{for } \; \alpha^s_t \geq \Psi/A_t \\ 0 & \text{for } \; \alpha^s_t < \Psi/A_t \end{cases} \tag{4.29}$$

In every period the household has to decide whether to adjust the housing stock to its optimal level, i.e. she has to determine whether the gains from correcting the housing stock justify paying the transaction costs $\lambda P_t H_{t-1}$.

[5] This constraint may seem redundant given that short sales are excluded in both stocks and bonds. However, $\alpha_s$ and $\alpha_f$ are also positive if both, stocks and bonds, are held in negative amounts in which case total financial investments become negative.

## 4.3 Calibration

I have not tried to estimate key parameters from original data myself. Instead, I rely on given estimates that are already more or less widely used in models that study portfolio selection over the life cycle. As most researchers concentrate on the U.S. my current calibration also uses estimates from that country. A natural extension of this thesis would repeat the calculations using German or even better multi-country OECD data.

**Time.** Households enter the model at age 20, work until age 65, and leave the model at age 80. Each period in the model is calibrated to correspond to five years which is done to reduce computation time. All parameters are adjusted accordingly.

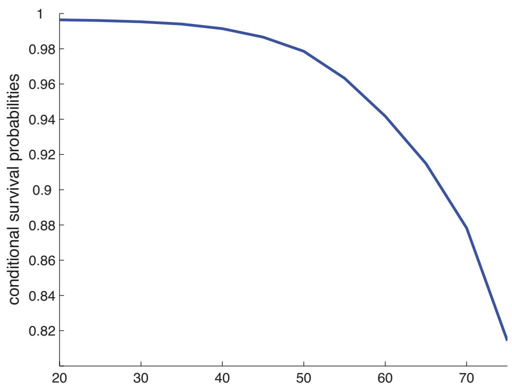
**Preferences.** In my baseline model I set the coefficient of relative risk aversion $\theta$ to 5 and the intertemporal discount factor $\beta$ to 0.95. The weight housing carries in the instantaneous utility function is set to 0.2 which approximately fits the average expenditure share accorded to housing in the 2001 Consumer Expenditure Survey.[6] The conditional survival probabilities $\pi_t$ are taken from United Nations (2000).

**Labor Income.** In my specification of the labor income process I follow Cocco, Gomes, and Maenhout (2005) who use a broad definition of labor income in order implicitly allow for endogenous ways of self insurance against pure labor income risk. They define labor income as the sum of reported labor income, unemployment compensation, social security, and other welfare and transfers such as help from relatives. The deterministic life-cycle wage profiles $f(t, Z_{kt})$ are taken from Altig, Auerbach, Kotlikoff, Smetters, and Walliser (2001) who fit a third degree polynomial for each of twelve groups which differ in the present discounted value of total lifetime earnings.[7] In the baseline model I use the wage profile of Altig et al.'s median group for all households. In a later variation I also study the effect of very low and very high

[6] Cf. Yao and Zhang (2005).

[7] Other studies such as Cocco, Gomes, and Maenhout (2005) use different wage profiles for groups with differing educational attainment.

**Figure 4.1:** Conditional survival probabilities




Note: Conditional survival probabilities for five year periods from United Nations (2000).

earnings. In particular I simulate the model using the wage profile of the second and the eleventh of Altig et al's twelve groups – i.e. the 2nd to 8th percentile and the 90th to 98th percentile of the income distribution.

The variance of the stochastic element of the labor income process is derived from the model of Campbell and Viceira (2001) and set to 0.0111. In their specification the stochastic component is split in permanent and transitory parts. However, on their way to the final values they first estimate the variance of the total innovation to detrended log income. That number fits to my specification.[8] As Campbell and Viceira (2001) distinguish between three educational levels I use the median value for the baseline case. In the variation with high and low earning processes I use the standard deviation of the high and low

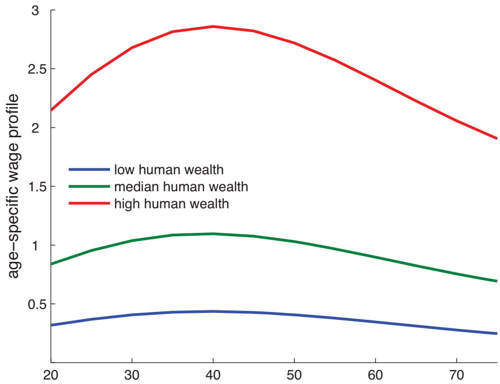
[8] Cf. Campbell and Viceira (2001), p. 208.

$$\log(Y_t)^* = \log(Y_t) - \hat{f}(t, Z_{kt}) \qquad \text{corresponds to } u_{kt}$$

$$\text{var}\big(\log(Y_t)^*\big) = \sigma^2_{\text{perm}} + 2\sigma^2_{\text{trans}} \qquad \text{corresponds to } \sigma^2_u$$

education group. In order to normalize the shock to one, I set the expected value to $-\sigma_u^2/2$.

**Figure 4.2:** Age-specific wage profile




Note: Age-specific wage profile for low (2nd - 10th percentile), median, and high (90th - 98th) expected discounted lifetime income groups taken from Altig, Auerbach, Kotlikoff, Smetters, and Walliser (2001).

There is a large literature on the results of nonzero correlation between income and return shocks. The general argument has already been presented in Section 3.1. However, in the remainder of this thesis I will concentrate on the effect of housing and abstract from a nonzero correlation of income and return shocks. During retirement, i.e. after age $\hat{T}$, the household receives the risk-free pension $\vartheta$. I set the pension to sixty percent of the household's expected income in the last year before retirement.

**Financial Assets.** I follow Cocco (2005) and parameterize the risk-free interest rate to two percent and the mortgage rate to four percent. In the data Cocco finds a higher mortgage premium of more than three percent for the period of 1964 to 1997, but notes that this premium overstates the applicable return. Part of the premium is paid for an implicit option on future inflation which is absent from my model as well as from his. Hence, I also set the mortgage premium to two percent. The excess return on stocks depends on two components. First, there is the expected excess return, the equity premium, that I set to four percent and hold fixed. That value is substantially lower than the historical average of 7-8 percent, but a widely used number in the literature.[9] Second, there is a stochastic element. The annual standard variation of the stochastic part is parameterized to 0.157 and thus equal to the historical value of the Standard & Poors 500 Index portfolio a number used e.g. by Campbell and Viceira (2001). The correlation of housing and stock returns is set to zero in the baseline model, but I try different values in later variations.

**Housing.** In the calibration of the house price process I follow again Cocco (2005) in order to make my results comparable to his. He reports a growth rate of log house prices of 0.016 per year and a standard deviation of the detrended log house prices of 0.062 which corresponds to a standard deviation of the housing return of 0.088.[10] Cocco uses self-assessed house values and finds a higher growth rate. However, part of that price increase is due to improvements in the quality of the property that cannot be accounted for in the data. In his and in my model those improvements would appear as housing investments and not as increases in the housing price. Hence, a lower growth rate is used. I also follow him in setting the annual depreciation rate of the housing stock one percent.

The transaction cost inherent in a house sale is set to eight percent and the down payment to twenty percent. For households who are older than 50 the credit constraint becomes tighter as lenders try to insure their loans against mortality risk. I chose a simple variant and increase the required down payment linearly until it reaches 1 at age 80. Those are key parameters for the model as they determine

[9] Cf. Fama and French (2002).

[10] The standard deviation of detrended log house prices $\sigma_\eta$ can easily be transformed into the standard deviation of the housing return $\sigma_{\Delta\eta}$ using the following formula.

$$\mathrm{var}(R^h) = \sigma^2_{\Delta\eta} = 2\sigma^2_\eta$$

important characteristics of the housing good, its illiquidity and its role as a collateral. Naturally, I also calculate specifications with more extreme values for the two parameters.

**Table 4.1:** Parameters in baseline model

| Parameter | Value |
|---|---|
| Entry age | 20 |
| Maximum age ($T$) | 80 |
| Retirement age ($\hat{T}$) | 65 |
| Curvature Parameter ($\theta$) | 5 |
| Discount factor ($\beta$) | 0.95 |
| Housing's weight in the utility function ($1-\phi$) | 0.2 |
| Risk-free return ($R^f$) | 0.02 |
| Mortgage rate ($R^d$) | 0.04 |
| Equity premium ($\mu$) | 0.04 |
| Growth rate of log house price ($g$) | 0.016 |
| Depreciation rate of housing stock ($\delta$) | 0.01 |
| Correlation of housing and stock returns ($\rho_{sh}$) | 0 |
| S.d. of idiosyncratic income shocks ($\sigma_u$) | 0.105 |
| S.d. of aggregate stock return shock ($\sigma_\epsilon$) | 0.157 |
| S.d. of aggregate detrended log house price ($\sigma_\eta$) | 0.062 |
| Minimum investment requirement ($\Psi$) | 0.1 |
| Transaction cost of house sale ($\lambda$) | 0.08 |
| Down payment until age 50 ($\kappa_{t\leq 50}$) | 0.20 |
| Down payment after age 50 ($\kappa_{t>50}$) | grows linearly to 1 |
| Pension benefits ($\vartheta_k$) | $0.6f(\hat{T}, Z_{k\hat{T}})$ |

Note: Figures in the table refer to annual values.

# Chapter 5

# Computation

## 5.1 Parameters and Shocks

**Transformation of Annual Parameters.** In order to reduce the computational burden I solve the model for 12 periods each of which corresponds to five years. Households enter the model in period 1 which lasts from age 20 to 24, retire after period 9 which lasts from age 60 to 64 and leave the model after period 12, i.e. just when turning 80. As a result, I have to transform most parameters from the annual values given in the calibration section to their five year equivalents. From now on, the hat refers to the transformed parameters.

The transformation is straightforward for parameters like $\beta$ and $R^d$ which are constant and not subject to any uncertainty. Hence, I use $\hat{\beta} = \beta^5$ and $\hat{R}^d = (1 + R)^{d\,5} - 1$. The conditional survival probabilities in the model are the product of the five corresponding annual probabilities, e.g. $\hat{\pi}_t = \prod_{i=5(t-1)+1}^{5t} \pi_i$ . The deterministic life cycle wage profile is found by simply taking the mean of the corresponding annual values, $\hat{f}(t, Z_{kt}) = 1/5 \sum_{i=5(t-1)+1}^{5t} f(i, Z_{ki})$ .

It is important to correct the stochastic shocks for the five-year periods. The asset return shock can be transformed by adjusting the standard deviation accordingly.[1] The very simple (bootstrap) simulation `transform_shocks.m` gives the following value for the baseline

[1] In the model the relevant stock return refers to an investor who adjusts her portfolio every five years.

case, $\hat{\sigma}_\epsilon = 0.0704$.[2]

The house price is determined by the underlying deterministic exponential trend and the i.i.d. shock. The standard deviation of the house price therefore does not change. The income process combines an age-specific wage profile and an idiosyncratic shock that is uncorrelated over time. For the average income shock over the five year horizon, the mean of the period income shocks, I find a standard deviation of $\hat{\sigma}_u = 0.0468$ using again the program `transform_shocks.m`.

**Approximation of Shocks.** I find the discretized distribution of the idiosyncratic income shock using the function `qnwnorm.m` that calculates nodes and weights for the approximation of normally distributed variables.[3] I first approximate the logarithm of the shock using five nodes and then calculate $\exp(u_{kt})$.

The house price is a state variable that I approximate with a deterministic exponential trend and a shock that follows a two-state Markov process which is calibrated to yield the standard deviation given above. The deterministic exponential trend in the process of house prices allows me to hold the size of the housing grid constant. This prevents a decline in the model's accuracy as households age. Imagine, for instance, that I let the house price follow a random walk. In that case the variance of house prices grows as the length of the horizon increases. Even if the possible realizations of the innovation are set to the minimum value of two, the difference between the largest and smallest possible house price grows over the life cycle and thereby reduces the model's accuracy at high age significantly for a given number of grid points.

The stochastic part of the stock return shock also features two possible realizations. The corresponding probabilities depend on the realization of the house price state which allows me to introduce a correlation between stock returns and house prices. There are altogether ten possible combinations of the return and income shock. Using a Kronecker product of the vectors that hold the individual weights I can easily combine those two sources of uncertainty in the program.

[2] The Matlab code for this function and the entire program are available from the author upon request, maxf@stanford.edu.

[3] The function is part of a toolbox by Paul L. Fackler and Mario J. Miranda.

## 5.2 Optimal Policy Functions

**Definition of the Grids.** The problem can only be solved using numerical techniques. Given the finite nature of the problem a solution exists and can be obtained by backward induction (Judd (1998)). The state variables cash-on-hand $X$ and last period's housing $H_{-1}$ are approximated with equally spaced grids, `x_grid` and `h_grid` with $n$ grid points each. The third state variable, the current housing price $P$, can only take on one of two values in every period. Hence, in each period I have to solve the problem for $n \times n \times 2$ possible combinations of the states.

I hold both the housing and the cash-on-hand grid constant. I arbitrarily set $H^{\min}$ equal to 0.01.[4] For the cash-on-hand grid I find the lowest possible value that a household's cash on hand can take on. This value that clearly makes the grid too large at the lower rim. However, it avoids errors in the simulation as I determined the grid size using approximated, discretized shocks. Extreme values in the simulation can otherwise lead the household to jump out of the specified grid. The maximum values for both grids were increased every time that a household in the simulation reached the upper end of the grid.

$$X^{\min} = Y^{\min} - (1 + R^d)(1 - \kappa^{\min}) P^{\max} H^{\max} \tag{5.1}$$

The control variables $C$, $D$, $I^H$, and $\alpha$ are also approximated with equally spaced grids with $m$ grid points each. Note that the grids for $D$ and $I^H$ are endogenous. The maximum housing disinvestment, $I^{H\ \min}$, depends on the minimum housing stock that I require the household to hold. I set the maximum housing investment exogenously and increased it each time the upper bound was found to be an optimal choice for the household. The mortgage can take on values between zero and the maximum that is reached when the credit constraint

[4] I cannot set the minimum house size to zero. The resulting utility would also be zero which cannot be transformed when I linearize the value function. Varying the minimum housing stock does does not affect the results as long as the value remains relatively small.

binds.

$$I_t^{H\ \min} = -H_{t-1} + H^{\min} \tag{5.2}$$
$$D_t^{\max} = (1-\kappa_t)P_t H_t\ . \tag{5.3}$$

For each combination of cash-on-hand and the housing stock I rule out the points at the lower end of the cash-on-hand grid that the households cannot possibly reach without violating a constraint. The condition is given by

$$X_t^{\min} \geq P_t I_t^{H\ \min} - D_t^{\max}\ . \tag{5.4}$$

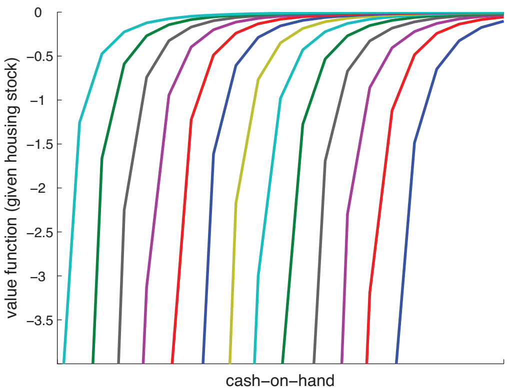
This allows to reduce the computational burden significantly, but reduces the number of grid points especially at low levels of the housing stock. In earlier versions I experimented with an alternative approach and employed an endogenously determined grid for cash-on-hand. However, the computational gain proved too small when weighted against the additional complication of the program.

For each state I have to calculate the solution twice, once under the assumption that the housing stock remains unchanged and once given that the household optimizes over all control variables while paying the applicable transaction cost. I then compare the resulting welfare and choose the superior alternative. That results in $n \times n \times 2 \times 2$ calculations.

**The Last Period.** I now summarize the solution technique in a few sentences before describing it in more detail below. In every period I calculate the utility associated with different choices of consumption, housing investments, mortgage, and the investment strategy for liquid assets. The value function is equal to the sum of the utility function and – in all but the last period – the expected discounted continuation value that is achieved given the chosen controls and the current states. The choices that are ruled out by the constraints are given a very large negative utility so that they are never chosen by the household. I optimize over the different choices and then iterate backwards.

I begin in the last period of the household's life, $T$, and calculate the utility she derives from optimally allocating her resources, $X_T$ and $H_{T-1}$, given the current house price $P_T$. As I abstract from a bequest motive and prevent the household from dying in debt I only need to find the optimal consumption and housing investment policy. I vectorize all possible combinations of $C_T$ and $I_T^H$ and calculate the resulting instantaneous utility which then allows me to find the optimal behavior of the household. Since this is the last period the utility function coincides with the value function, $u(C_T, H_T) = \mathcal{V}_T$. I then compare the two solutions, for a fixed and for an adjusted housing stock, and store the optimal policies and the resulting value function.

**Figure 5.1:** Value function



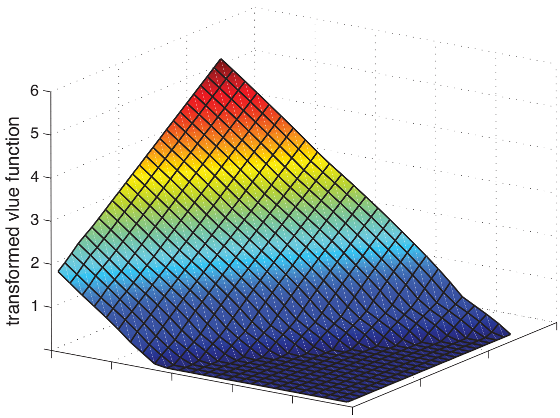

Note: Value function in the oldest age bracket for different values of the housing stock given that the house price takes on the low realization. $\mathcal{V}_T^j(X_T, H_{T-1}^j; P_T^{\text{low}})$

For each realization of the house price the value function is highly concave as is more than evident in Figure 5.1. I need to interpolate on the value function when calling on the continuation value in the optimization in the previous period. In order to increase the accuracy of the interpolation I linearize the value function similar to Carroll (2002a) and find $\Lambda_T$. The transformation is imperfect as it ignores the presence of housing. However, given the relatively small role housing plays in the utility function, the large $\phi$, the results seem to be reasonably accurate. The transformed value function is shown in Figure

5.2 which clearly allows a more accurate linear interpolation.[5] The flat region in the graph corresponds to combinations of the states that the household cannot reach without violating a constraint that follows from (5.4).

$$\Lambda_T(X_T, H_{T-1}; P_T) = -\mathcal{V}_T(X_T, H_{T-1}; P_T)^{\frac{1}{1-\theta}} \tag{5.5}$$

**Figure 5.2:** Transformed value function




Note: Transformed value function for the oldest age bracket on the two-dimensional grid spanned by `x_grid` and `h_grid` given that the house price takes on the low realization. $\Lambda_T(X_T, H_{T-1}; P_T^{\text{low}})$

**Backward Induction.** In the next to last period the household optimizes over all control variables knowing the expected continuation value for all points on the two grids. For all combinations of the states I again vectorize the problem. For each point on the grid for housing investments, `hinv_grid`, the new housing stock $H_{T-1}$

[5] Due to the specification of the utility function both utility and the value function are negative. Therefore, I have to multiply $\mathcal{V}$ with $-1$ before the transformation.

and the maximal mortgage $D_{T-1}^{\max}$ is calculated which then allows to span the `d_grid`. For each combination of the two grids I append the savings grid, `s_grid`, and calculate the resulting levels of consumption which are simply the resources left-over in $X_{T-1} + D_{T-1} = C_{T-1} + P_{T-1} I_{T-1}^{H} + A_{T-1}$. In a final step, the grid for the allocation of liquid assets among stocks and bills, `alpha_grid`, has to be added.[6]

The law of motion for cash-on-hand is then applied to each of the resulting combinations of controls and all ten possible realizations of the income and return shocks to find the associated values of next period's cash-on-hand, $\tilde{X}_T^i$, for $i = 1, ..., m \times m \times m \times m \times 10$.[7]

It has been illustrated above that the stochastic element of the house price process is approximated with a two-state Markov process. I therefore loop over both possible future values of the process and find the continuation value of $*_T$ on the applicable grid for each value of $\tilde{X}_T^i$ and the corresponding value of $H_{T-1}^i$. In between the grid points I use the standard function `interp2.m` and interpolate linearly. Of course, the transformation of the value function needs to be reversed before the household's expectations over the income and return shocks can be calculated. The expectations can then be found by multiplying with the vector of the combined weights.[8]

$$\mathbb{E}_{Y,R^s}\left[\mathcal{V}_T(\tilde{X}_T^i, H_{T-1}^i; \tilde{P}_T)\right] = \mathbb{E}_{Y,R^s}\left[-\Lambda_T(\tilde{X}_T^i, H_{T-1}^i; \tilde{P}_T)^{1-\theta}\right] \quad (5.6)$$

The current value function can be found as the sum of instantaneous utility and the expected discounted continuation value where the expectation now also refers to the uncertain shock to the house price. Again the expectation is easy to derive by multiplying with the applicable row of the Markov transition matrix for the house price.[9]

$$\begin{aligned}\mathcal{V}_{T-1}^i(X_{T-1}, H_{T-2}; P_{T-1}) = \\ u(C_{T-1}^i, H_{T-1}^i) + \hat{\beta}\hat{\pi}_{T-1}\mathbb{E}_{Y,R^s,P}\left[\mathcal{V}_T(\tilde{X}_T^i, H_{T-1}^i; \tilde{P}_T)\right] \quad (5.7)\end{aligned}$$

[6] In the Matlab code the combinations of the control variables are stored in the matrix `X` where the columns 1 to 6 hold the housing investment, the mortgage taken, the investment in liquid assets, consumption, the share of stocks, and the new housing stock.

[7] The associated values are stored in the vector `Z`.

[8] The expected continuation values conditional on next period's house price are stored in the matrix `Q`.

[9] The matrix `Y` holds the current utility in column 1, the expected continuation value in column 2, and the current value in column 3.

The optimal policy for each admissible combination of the states refers to the combination of controls $i$ that yields the highest welfare in period $T-1$.

$$\mathcal{V}_{T-1}(X_{T-1}, H_{T-2}; P_{T-1}) = \max_i \{\mathcal{V}^i_{T-1}(X_{T-1}, H_{T-2}; P_{T-1})\} \quad (5.8)$$

The corresponding values of the control variables and the value function are again stored on the grid spanned by `h_grid` and `x_grid`. The same is done with the value function; however, the value function first needs to be transformed as in (5.5). I then iterate backwards and solve all previous periods in the same way.

The optimal policies for the baseline model have been calculated using 25 grid points for the two states and 13 grid points for the control variables. Together with the ten possible realizations of the discretized income and asset return shock this results in $15^4 \cdot 10 = 285,610$ calculations at $25^2 \cdot 2 = 1250$ combinations of the state variables for each period. Altogether the computer runs for about eleven hours.

## 5.3 Simulation

This model concentrates on the demand of households for housing and other assets over the life cycle. It abstracts form general equilibrium aspects and does not incorporate any interaction of the households in the model. The housing price, for instance, is taken as given and not affected by the households allocation decisions. Hence, I do not have to simulate multiple cohorts, but only follow one household at a time.

Having calculated the optimal policy functions it is straightforward to simulate the behavior of individual households over the life cycle. For each run of the simulation I draw realizations of the two aggregate shocks, the asset return shock and the house price shock are identical for all households in the simulation run. The idiosyncratic shocks, though, are drawn individually for each household.

In the baseline model the household begins her adult life endowed with a house of the minimum size but without any other resources in the form of liquid assets. However, she does dispose of her current labor income. Given the state variables, cash-on-hand which equals the labor income in period one, the minimum housing stock and the realization of the aggregate house price the households behavior can be found by calling on the optimal policy functions for consumption, housing investments, debt, and asset allocation. This is again done by two dimensional interpolation on the grid spanned by `x_grid` and `h_grid`. Next period's cash-on-hand and housing stock are calculated using the laws of motion and the realizations of the shocks drawn above. With those at hand the process can be repeated for the next and all the following periods. Altogether I ran 20 simulations with 50 households each.

# Chapter 6

# Simulation Results

In this part I present and interpret the results of the simulation that uses the baseline parameter constellation as introduced in Section 4.3. After this I will highlight main results of alternate models that vary slightly from this baseline case. The aim is to identify the key features of the model, those features that eventually drive the results.

Heaton and Lucas (2000) use three definitions of households' net worth: 'liquid net worth', the sum of stocks and bonds minus debt; 'financial net worth', liquid net worth plus the housing stock's value; and 'total net worth', the sum of financial net worth and human wealth. Without the subtraction of debt the classifications refer to 'liquid assets', 'financial assets', and 'total assets'.

## 6.1 The Baseline Model

One can easily calculate a household's human wealth given the age-specific wage profiles and the assumptions concerning the income process and the public pension system. As before human wealth is simply the expected present discounted value of future labor income and pension benefits. Following again Heaton and Lucas (2000) a constant annual discount rate of five percent is applied.[1] Figure 6.1 illustrates human wealth over the life cycle for the median income group. Fig-

[1] A more precise calculation requires a risk-adjusted discount rate as the pension benefit is risk-free whereas the labor income stream conveys an idiosyncratic risk term.

ure 6.2 shows the average realized paths of labor income and pension benefits for several runs of the simulation. Not surprisingly, the idiosyncratic component of labor income almost cancels out over the 50 households.

Households' average spending on the nondurable consumption good increases with age and seems to jump up in the last period of life. This surprising finding of Figure 6.3 can best be understood when considered together with Figure 6.7. In addition to their nondurable consumption households derive housing services from their durable housing stock. That stock is high early in life, subsequently depreciates, and falls significantly during retirement. The reason can be seen in housing's dual role. There are basically two incentives for holding housing stock: the consumption and the investment motive. Late in life, housing loses its role as an investment good until, in the very last period, it is just another consumption good that the household can spend her resources on. Hence, the household shifts resources away from housing and into non-durable consumption late in life.[2]

As bequests are absent here households enter the model with zero levels of housing stock, debt, and financial investments, but equipped with their first labor income. Early in life, the two motives for holding housing stock imply high housing investments (Figure 6.5). Adjustments of the housing stock are infrequent for most part of the households' lifespan as they are costly. Figure 6.6 shows that less than five percent of the households move in each period between age 35 and 60. Significant disinvestments do not occur before retirement when the households begin do run down their financial savings as well as their housing stock in order to finance nondurable consumption in excess of their pension benefits. Average cash-on-hand (Figure 6.4) is relatively high in the model's starting period and then falls due to the investments in housing and the mortgage payments, but later rises when the household accumulates more wealth.

Figures 6.7 and 6.8 illustrate the life cycle pattern of the absolute housing stock and the housing stock's value respectively. While the housing stock is built up early in life, it then remains practically untouched and runs down as it is subject to depreciation. The value

[2] Of course, other arguments can be put forward for the decrease in average house sizes among the elderly. A brief discussion can be found in Chapter 7.

of the stock, however, increases over time as the house price's trend growth exceeds the depreciation rate. As already stated above, the housing stock is reduced late in life.

Households take out high mortgages when they are young in order to finance the investment in housing. Accordingly, about half of the households face a binding credit constraint in the first period. Debt levels (Figure 6.9) remain high – when the formerly credit constrained households adjust their mortgage in the subsequent periods – and fall later in life. The share of credit constrained households (Figure 6.10) increases again after retirement when the credit constraint becomes tighter as specified in the calibration section. Figure 6.11 shows savings that are invested into liquid assets, i.e. stocks and bonds. A clear hump shape is more than evident.

Figure 6.12 and 6.13 show average asset holdings and net worth for all three definitions introduced above. Note that the liquid net worth positions remain negative for the first half of most households' lifespan. Financial and total assets include the housing stock and thus do not fall to zero in the last period.

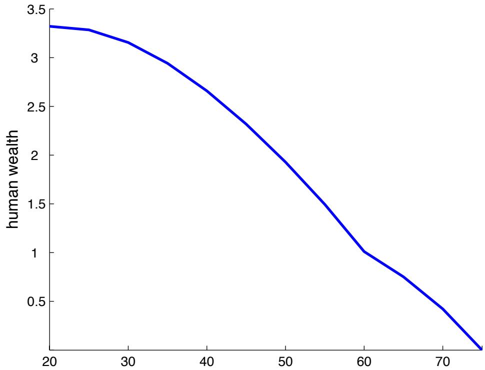
While average stock holdings also follow a clear hump shape, bonds are only held during the first and the last periods. Interestingly, this result is mainly driven by the stock market participation decision (Figure 6.16) as households who do participate in the stock market generally accord all their savings to risky stock investments. Many young investors, however, spend most of their available resources on housing investments and thus do not reach the threshold introduced by the minimum investment requirement. Similar to the 'housing constraint' in Flavin and Yamashita (2002) the importance of the housing stock relative to financial assets decreases over the life cycle and only increases late in life when households run down their total financial assets while keeping part of their housing stock because of the consumption motive.

In Cocco's model (2005) the participation rate does not fall later in life – a difference that is due to the fixed cost of participation that, once payed, allow investors access to the stock market for the remainder of their lives. Here, I instead assumed a minimum investment requirement that prevents very low stock investments without imposing a direct monetary or a utility cost. The minimum investment requirement becomes again binding for a large share of the investors during retirement when financial savings are run down.

A second reason for lower risk taking in the financial portfolio at high age results from the human wealth argument presented in chapter 2. Capitalized labor income is the dominant component of total wealth for the young (Figure 6.19), but less important for older investors whose tilt in the financial portfolio toward risky assets is thus lower.[3]

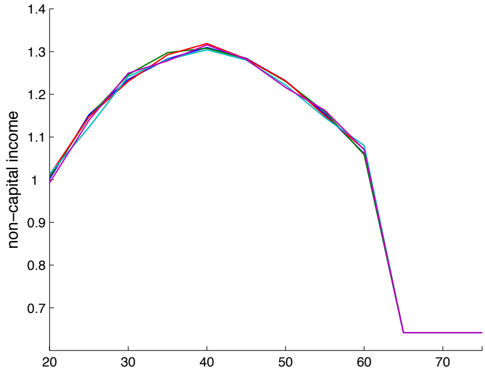
[3]A further motive for lower risky asset shares during retirement can be seen in the higher business risk faced especially by relatively old and wealthy investors. This, however, is absent from my model. See Heaton and Lucas (2000).

**Figure 6.1:** Human wealth



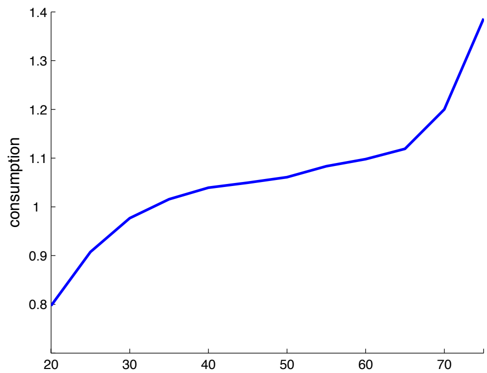

Note: Capitalized labor income and pension benefits for the median income group using a constant discount rate of five percent.

**Figure 6.2:** Labor income and pension benefits



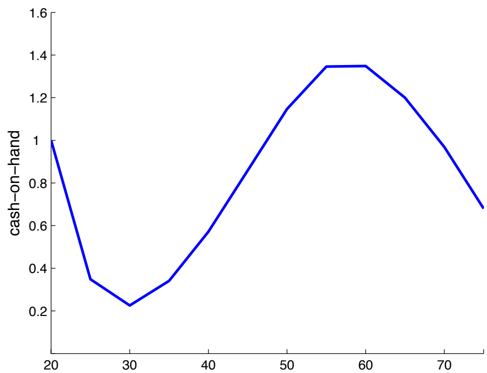

Note: Noncapital income, i.e. labor income streams and pension benefits, normalized to one in the first period.

**Figure 6.3:** Consumption



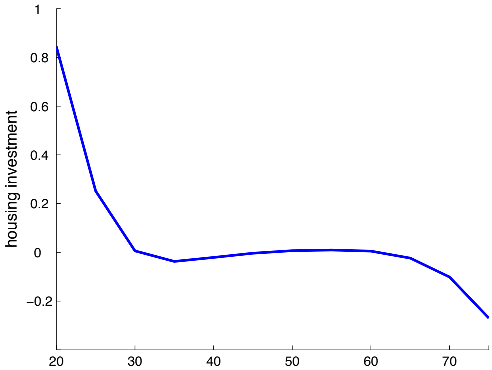

Note: Average simulated consumption normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.4:** Cash-on-hand



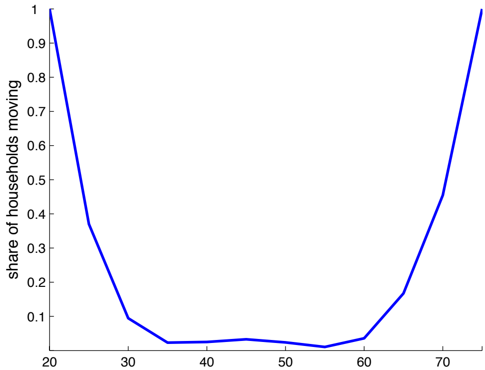

Note: Average simulated cash-on-hand normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.5:** Housing investment



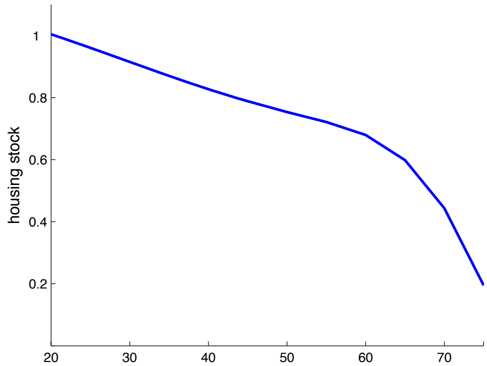

Note: Average simulated gross housing investment normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.6:** Share of households moving



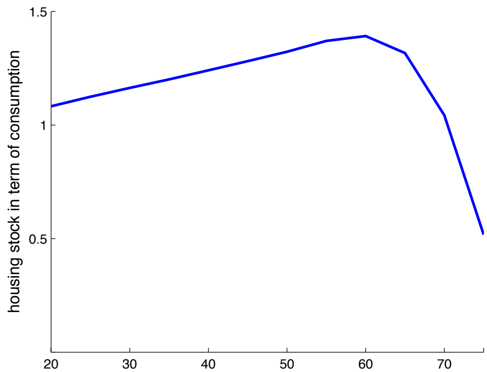

Note: Average simulated share of households that adjust their housing stock in each period.

**Figure 6.7:** Housing stock



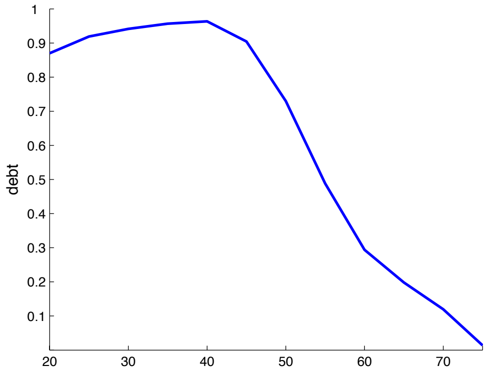

Note: Average simulated absolute housing stock normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.8:** Value of the housing stock



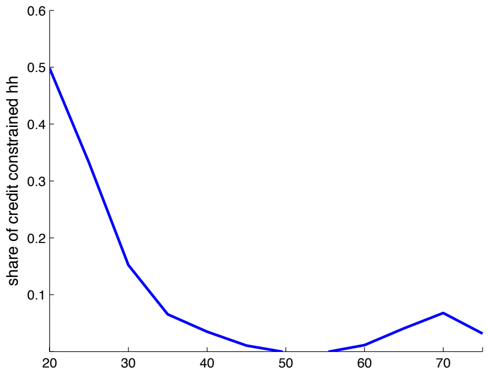

Note: Average simulated housing stock in terms of the consumption good normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.9:** Debt



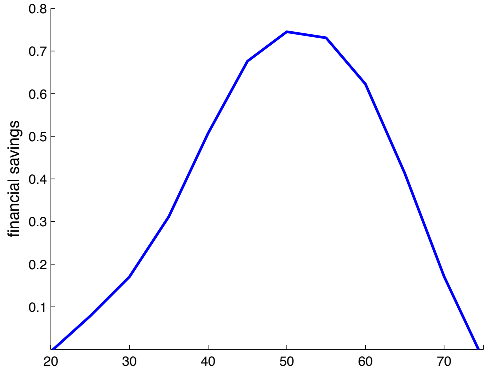

Note: Average simulated mortgage normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.10:** Share of credit constrained households



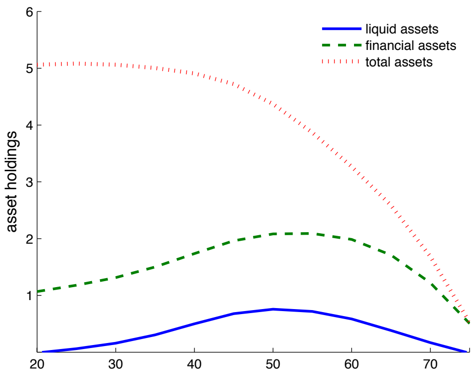

Note: Average share of households that are credit constrained in each period.

**Figure 6.11:** Savings



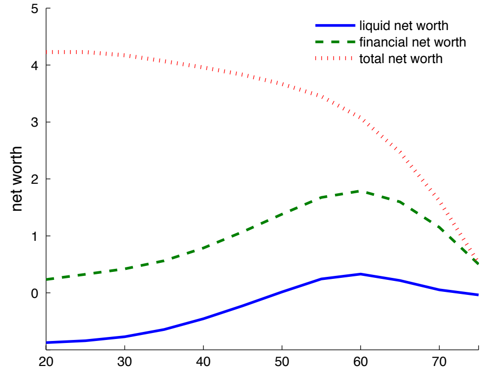

Note: Average savings ($A$) that are invested in stocks or bonds normalized by the expected labor income in the first period.

**Figure 6.12:** Asset holdings



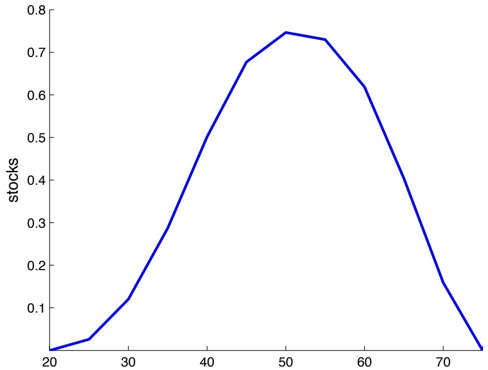

Note: Average holdings of liquid ($A$), financial ($A + PH$), and total assets ($A + PH + HW$) normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.13:** Net worth



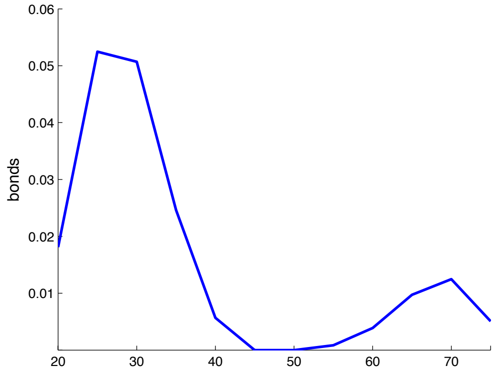

Note: Average liquid $(A - D)$, financial $(A + PH - D)$, and total net worth $(A + PH + HW - D)$ normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.14:** Stock holdings



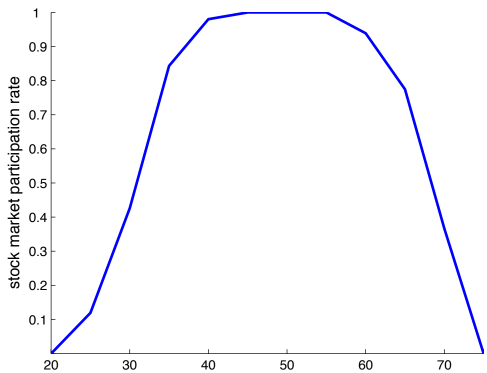

Note: Average stock holdings normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.15:** Bond holdings



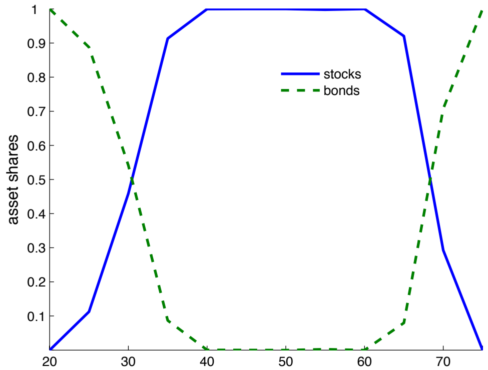

Note: Average bond holdings normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.16:** Stock market participation rate



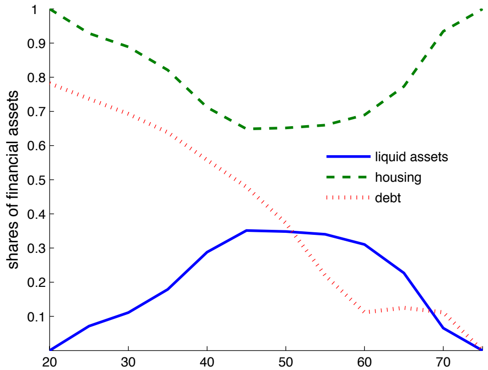

Note: Share of households that participate in the stock market in each period.

**Figure 6.17:** Portfolio allocation – liquid assets



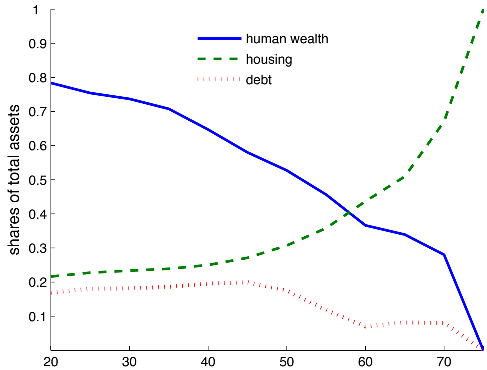

Note: Allocation of liquid assets among stocks and bonds in each period.

**Figure 6.18:** Portfolio allocation – financial assets




Note: Allocation of financial assets among liquid assets, housing, and debt in each period.

**Figure 6.19:** Portfolio allocation – total assets




Note: Human wealth, housing, and debt as shares of total assets in each period.

## 6.2 Alternative Models

**Credit Constraint.** In order to illustrate the role played by the mortgage contract I set the parameter $\kappa$ equal to one in an alternative specification. This implies a down payment of one hundred percent or – as households cannot sell bonds short – the plain absence of debt. In the baseline model households generally acquire a relatively large house early in life in order to reduce the number of costly future house trades. Here, they cannot partially finance that investment with a mortgage. Hence, housing investments are lower in the first period, but the average adjustment does not drop down to zero by age 30 as in the baseline model. Instead, households continue to accumulate housing stock until age 50 as can be seen in Figure 6.20. As a corollary, the share of moving households remains high for much longer. Households are prevented from borrowing against future labor income or the housing stock for financing not only housing investments, but also stock and bond investments. This results in significantly lower financial investments which Figure 6.21 illustrates. Accordingly, many investors reach the minimum investment later than before. The lag in stock market participation is evident in the life cycle pattern (Figure 6.22).

To the other extreme I simulate the model without a down payment requirement which allows households to borrow up to the value of their house. Not surprisingly, the simulation yields results that amplify the difference between the 'no debt financing' and the baseline case. Households immediately choose the desired house size in the first period as they are unconstrained. Financial savings are higher and less investors fail to reach the minimum investment requirement.

**Minimum Investment Requirement.** I have already elaborated on the interaction between housing, the credit constraint, and the minimum investment requirement. Young homeowners have are credit constrained and show low financial savings due to their investments in the housing stock. As a result, many do not reach the minimum investment requirement and thus do not hold stocks at all. In an alternate calculation I have solved the identical model without costs of asset market participation. The results illustrate the important role by the this market friction. While savings remain approximately unchanged, investors now hold all their liquid assets in stocks a prediction that is clearly at odds with the evidence. Stockholdings double among the young and they are still 25 percent higher at age 45. Stocks form a significantly larger part of financial assets during the first half of households' lives (6.23).

**Transaction Costs in Housing Market.** Another variation of the baseline model features housing as a liquid investment good. While houses are still traded as a whole, there are no longer transaction costs that the seller incurs. It is reasonable to expect that housing becomes more attractive in this setting. In the first period, however, households now invest slightly less in the housing stock. In the baseline model the housing in the first period not only reflects the consumption and investment motive, but also the attempt to reduce the number of future house trades. By acquiring a house that is 'too big', young households prevent subsequent, costly adjustments when their desire for housing grows along with their wealth. Here, those adjustments are free and the average housing stock consequently grows until age 45 and is significantly larger than in the baseline model (Figure 6.24). At the same time, households now hold more leveraged portfolios and achieve levels of financial net worth that exceed those in the baseline case by about ten percent.

**House Risk.** In order to study the effects of house price risk on household portfolios I also solve the model with a deterministic housing price. Contrary to my own expectation the average housing stock does not rise which might at first seem to be a puzzle. Note, however, that housing is an important share of total wealth especially late in life. Hence, an uncertain housing price can have very strong effects among felicity among the elderly. Now, without that source of risk, households no longer need to hold extra high levels of housing stock in order to cushion the possible losses. Accordingly, consumption rises among the young who now save less, both in terms of liquid assets and housing investments. Investors reach the minimum investment requirement later than before and absolute stockholdings are lower (Figure 6.25). At first sight, this prediction stands in stark contrast with Cocco's finding that house price risk crowds out stockholdings. In the 'no house price risk' variant of his model he reports a higher share of financial assets that is allocated to stocks. This also holds in my model. However, the share only rises because of the significant decline in financial assets, the denominator (Figure 6.26. This does not appear in Cocco's presentation.

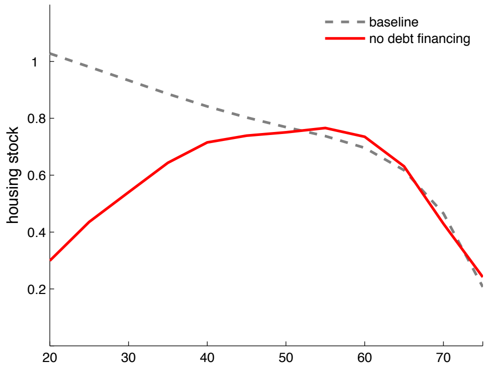
**Income Risk.** An increase in idiosyncratic labor income risk does not yield any surprising results. Households now save more when young in order to insure themselves against negative income shocks. When approaching retirement the effect reverses. Pension benefits are risk free such that the setting then resembles the baseline case more and more. The higher savings in earlier periods now allow higher levels of consumption and lower savings (6.27).

**Stock Return Risk.** Stock investments become less attractive when I increase the standard deviation of returns. Hence, households save less of their resources for financial investments, they participate in the stock market later in life and thus accumulate wealth slower. Stockholdings decrease (Figure 6.28 and the financial net worth is lower especially in the second half of their lives.

**Return Correlation.** In the baseline model the correlation between stock and housing returns is zero. This assumption corresponds to the finding of, for instance, Flavin and Nakagawa (2004).[4] When simulating the model with a positive correlation I find that asset demand decreases. Stocks no longer provide a reasonable hedge for the house price risk and become less attractive investments. While the share of financial assets that is accorded to stocks hardly falls at all, there is a significant drop in absolute stockholdings (6.29). The reduced absolute level of financial assets again hides the effect when only the portfolio composition is considered.

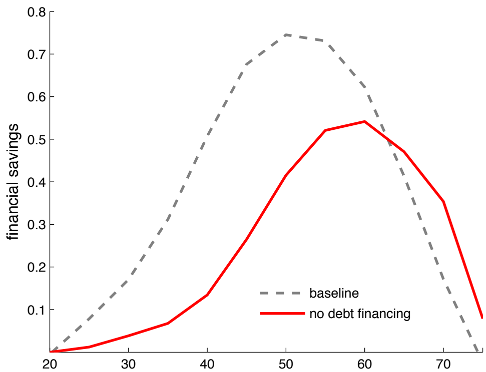
[4]Other researchers highlight the possible positive correlation between housing prices and income shocks which emphasis on regional correlation. Unfortunately, I cannot easily include this in my model.

**Figure 6.20:** Housing stock – 'no debt financing'



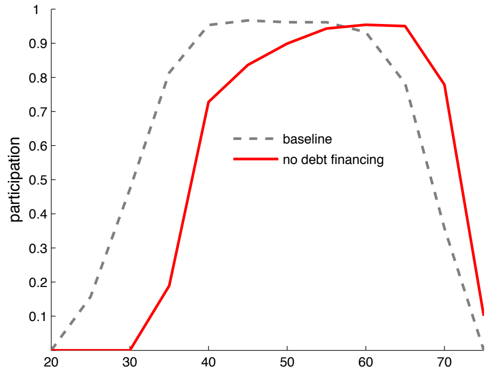

Note: Average simulated absolute housing stock normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.21:** Financial savings – 'no debt financing'



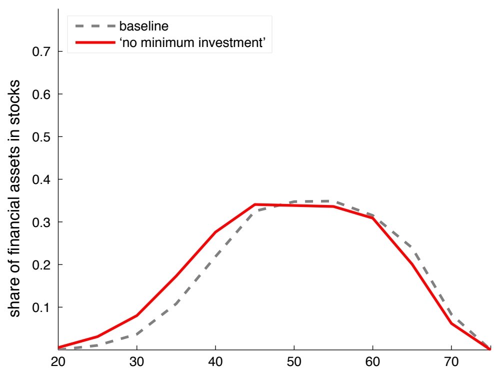

Note: Average savings ($A$) that are invested in stocks or bonds in each period normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.22:** Stock market participation– 'no debt financing'



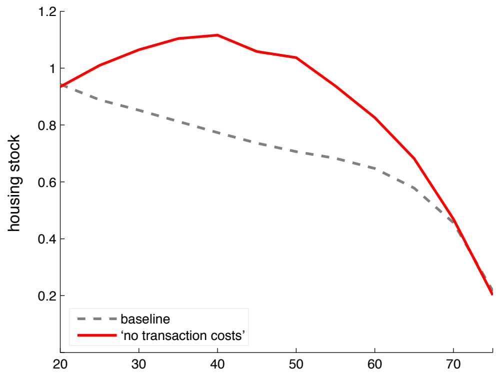

Note: Share of households that participate in the stock market in each period.

**Figure 6.23:** Stocks as share of financial assets – 'no minimum investment requirement'



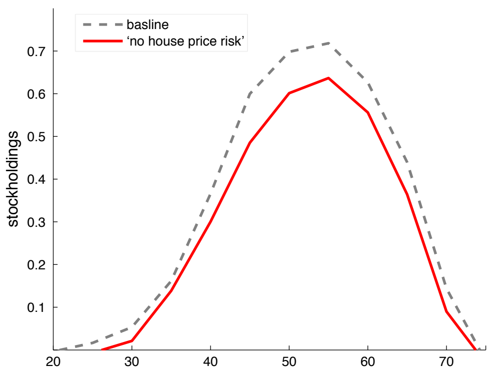

Note: Stocks as share of financial assets in the baseline model and in the variation without a minimum investment requirement.

**Figure 6.24:** Housing stock – 'no transaction costs'



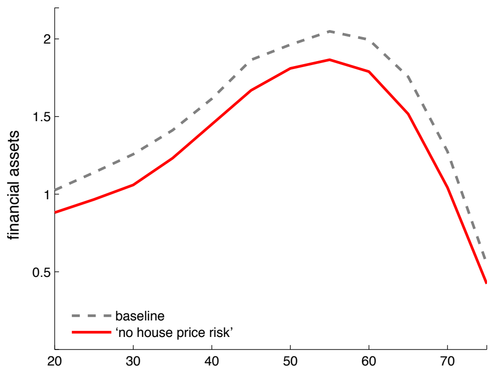

Note: Absolute housing stock in the baseline model and the variation with costless house trades normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.25:** Stockholdings – 'no house price risk'



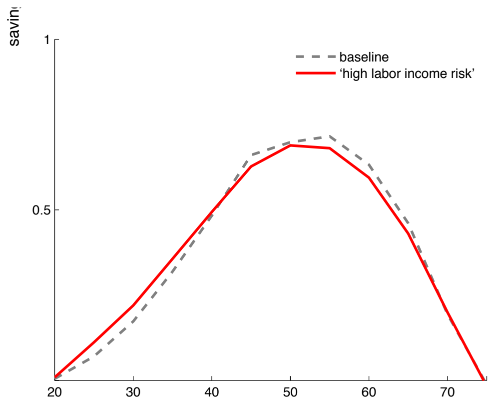

Note: Stockholdings in the baseline model and in the variation without house price risk normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.26:** Financial Assets – 'no house price risk'



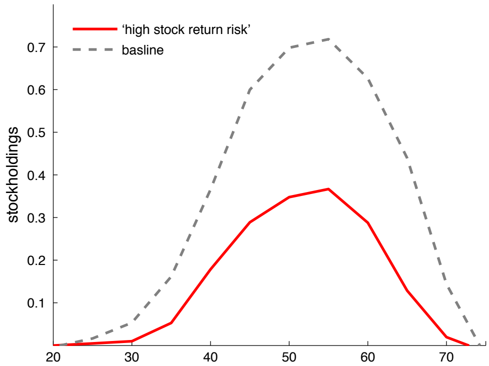

Note: Financial assets in the baseline model and in the variation without house price risk normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.27:** Savings – 'high labor income risk'



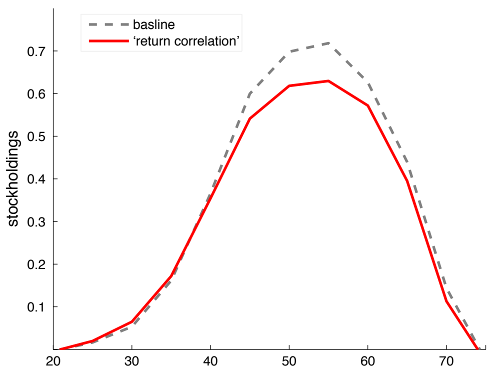

Note: Savings in the baseline model and in the variation with high labor income risk normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.28:** Stockholdings – 'high stock return risk'




Note: Stockholdings in the baseline model and in the variation with high stock return risk normalized by the expected labor income in the first period, i.e. at age 20-25.

**Figure 6.29:** Stockholdings – 'return correlation'




Note: Stockholdings in the baseline model and in the variation with positive correlation between assets returns and the housing price normalized by the expected labor income in the first period.

# Chapter 7

# Shortcomings and Concluding Remarks

**Bond Holdings.** While the model manages to reproduce the general life cycle pattern of observed stock market participation rates it does not predict realistic portfolio shares conditional on asset market participation. In the model, interest payments on the mortgage exceed the risk free return. Hence, no investor is willing to hold debt and bonds at the same time as she can renegotiate her mortgage level at no cost. Cocco (2005) discusses two possible extensions: first, adjustment costs for the mortgage level might be introduced into the model. The investor would then be induced to accommodate small variations in wealth by adjusting her liquid bond holdings. However, that strategy requires the introduction of outstanding debt as an additional state variable with significant negative impact on computation speed. An alternative approach supposes that investors need to hold minimum levels of liquid bonds for transaction purposes. The minimum levels might for instance be related to lagged nondurable consumption. While the second strategy is computationally more feasible, it requires ad-hoc parameterization of the liquidity need.

**Voluntary and Forced Moving.** In the model presented here adjustments of housing stock are voluntary and endogenous. However, there are good reasons for assuming exogenous shocks that have an impact on the households' desired housing stock. First, the demand for housing stock most certainly depends on the household size which varies over the life cycle. Critical events might include the marriage, children who suddenly appear and then disappear again after two decades, possibly a divorce or the death of a family member. Second, health related shocks can not only affect people's earning abilities, but also their demand for housing: permanent illness might, for instance, force elderly investors to permanently live in an old people's home, also reducing the demand for housing stock. Moreover, investors who downsize from larger to smaller homes due to an exogenous shock now need to allocate the proceeds among stocks and bonds. Those aspects are absent from my model, though.

I have argued in this thesis that the coexistence of financial and real assets can have significant impact on optimal household portfolios. Liquidity constraints interact with an exogenous minimum investment requirement and show potential to explain observed asset market participation patterns. For the young household, the desire to own a house depresses financial savings. This effect is even stronger in the presence of transaction costs in housing market: young households now anticipate future increases in wealth, and thus housing demand, which leads to even higher housing investments in the first periods. Less households reach the minimum investment requirement early in life – participation rates among the young are lower.

# Appendix A

# Results of Portfolio Choice Theory

## A.1 Appendix to Section 2.1

The housing stock is fixed its current level. The portfolio share of housing, the ratio of the housing stock to total wealth, is thus fixed to $\alpha_h = \bar{\alpha}_h$. The investor's problem is

$$\begin{aligned} \min_{\boldsymbol{\alpha}} \boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha} \quad & \text{subject to} \\ & \boldsymbol{\alpha}'\boldsymbol{\mu} + R^f = \bar{R} \\ & \alpha_h = \bar{\alpha}_h. \end{aligned} \tag{A.1}$$

One can define the Lagrangian and calculate the first order conditions. Note that the subscript $-h$ identifies the same vector as before except for the missing last element that concerns housing. In order to highlight the connection with the coefficient of relative risk aversion the Lagrange multiplier is set to $2\theta$.

$$\mathcal{L} = \boldsymbol{\alpha}'_{-h}\boldsymbol{\Sigma}\boldsymbol{\alpha}_{-h} + \alpha_h^2\sigma_h^2 + 2\alpha_h\boldsymbol{\alpha}'_{-h}\boldsymbol{\Gamma}_{sh} - \frac{2}{\theta}\left(\boldsymbol{\alpha}'_{-h}\boldsymbol{\mu}_{-i} + \alpha_h\mu_h + R^f - \bar{R}\right) \tag{A.2}$$

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{\alpha}'_{-h}} = \boldsymbol{\Sigma}\boldsymbol{\alpha}_{-h} + \alpha_h\boldsymbol{\Gamma}_{sh} - \frac{1}{\theta}\boldsymbol{\mu}_{-h} = 0 \tag{A.3}$$

Solving for the optimal portfolio shares leads to the well known solution $\forall i$,

$$\alpha_i = \mathbf{\Sigma}^{-1}\frac{\mu_i}{\theta} - \alpha_h \mathbf{\Sigma}^{-1}\mathbf{\Gamma}_{sh} \,. \tag{A.4}$$

## A.2 Appendix to Section 2.2

In this section which draws on the appendix to (Campbell and Viceira 2002) I will show that $\boldsymbol{\alpha}^{opt}$ indeed maximizes $\mathbb{E}\left[(1+R^p)^{1-\theta}\right]$ in our value function under the assumption of CRRA utility and jointly log-normally distributed returns for equity and housing investments. As preliminary step I will first derive an expression for $\log(1+R^p) = r^p$. In this section I depart from the previous notation in that I use the symbol $\boldsymbol{\Sigma}$ for the variance-covariance matrix of all returns as housing is just an additional asset without any special feature in this model.

$$\begin{pmatrix} R^s \\ R^h \end{pmatrix} \sim LN\left(\begin{bmatrix} \mu_s \\ \mu_h \end{bmatrix}, \boldsymbol{\Sigma}\right) \tag{A.5}$$

**Finding an Expression for $r^p$.** As already mentioned, the total portfolio return depends on the portfolio shares $\boldsymbol{\alpha}$ and the returns of equity and housing investments which are stacked into the vector $\boldsymbol{R}$.

$$R^p = R^f + \boldsymbol{\alpha}'(\boldsymbol{R} - R^f \boldsymbol{\iota}) \tag{A.6}$$

One can modify this equation in order to find an expression for $r^p - r^f$ as a function of $\boldsymbol{r} - r^f \boldsymbol{\iota}$ and the portfolio shares. I will call this function $G$.

$$\frac{1+R^p}{1+R^f} = 1 + \boldsymbol{\alpha}' \begin{bmatrix} \frac{1+R^s}{1+R^f} & -1 \\ \frac{1+R^h}{1+R^f} & -1 \end{bmatrix} \tag{A.7}$$

$$r^p - r^f = \log\left(1 + \boldsymbol{\alpha}'\left(\exp(\boldsymbol{r} - r^f\boldsymbol{\iota}) - 1\right)\right) = G(\boldsymbol{r} - r^f\boldsymbol{\iota}) \tag{A.8}$$

A second-order Taylor expansion around the point $r^s = r^h = r^f$ leads to

$$\begin{aligned} G(\boldsymbol{r} - r^f\boldsymbol{\iota}) \quad \approx \quad & G(0) + G'|_0(\boldsymbol{r} - r^f\boldsymbol{\iota}) \\ & + \frac{1}{2}(\boldsymbol{r} - r^f\boldsymbol{\iota})'\, G''|_0(\boldsymbol{r} - r^f\boldsymbol{\iota}) \end{aligned} \tag{A.9}$$

where

$$G(0) = 0 \tag{A.10}$$

$$G'|_0 = \frac{\delta G}{\delta \boldsymbol{\alpha}} = \begin{pmatrix} \alpha_s \\ \alpha_h \end{pmatrix} \tag{A.11}$$

$$G''|_0 = \frac{\delta^2 G}{\delta \boldsymbol{\alpha}\, \delta \boldsymbol{\alpha}'} = \begin{pmatrix} \alpha_s(1-\alpha_s) & -\alpha_s\alpha_h \\ -\alpha_s\alpha_h & \alpha_h(1-\alpha_h) \end{pmatrix} . \tag{A.12}$$

Thus,

$$\begin{aligned} G(\boldsymbol{r} - r^f \boldsymbol{\iota}) \approx \alpha_s(r^s - r^f) + \alpha_h(r^h - r^f) \\ + \frac{1}{2}\,\alpha_s(1-\alpha_s)\sigma_s^2 + \frac{1}{2}\,\alpha_h(1-\alpha_h)\sigma_h^2 - \alpha_s\alpha_h\sigma_{sh} \,, \end{aligned} \tag{A.13}$$

which in matrix notation takes on the nicer form

$$G(\boldsymbol{r} - r^f \boldsymbol{\iota}) \approx \boldsymbol{\alpha}'(\boldsymbol{r} - r^f \boldsymbol{\iota}) + \frac{1}{2}\,\boldsymbol{\alpha}'\boldsymbol{\sigma}^2 - \frac{1}{2}\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha} \,. \tag{A.14}$$

The elegant expression that has been derived for $r^p$ will simplify the optimization problem below.

$$r^p \approx r^f + \boldsymbol{\alpha}'(\boldsymbol{r} - r^f \boldsymbol{\iota}) + \frac{1}{2}\,\boldsymbol{\alpha}'\boldsymbol{\sigma}^2 - \frac{1}{2}\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha} \tag{A.15}$$

**Optimization.** Maximizing the last part of the value function above, $\mathbb{E}\left[(1+R^p)^{1-\theta}\right]$. Obviously, the problem is equivalent to maximizing $\log \mathbb{E}\left[(1+R^p)^{1-\theta}\right]$. Note that I assumed jointly lognormally distributed returns on the risky equity and housing investments. This allows the use of a very general property. Let $X \sim LN$ such that $\log X$ follows a normal distribution with $\log X = x \sim N(\mu, \sigma^2)$. Then,

$$\begin{aligned} \mathbb{E}\, X &= \exp\left(\mu + \frac{1}{2}\,\sigma^2\right) \\ \log \mathbb{E}\, X &= \mathbb{E}\, \log X + \frac{1}{2}\,\mathrm{var}\,(\log X) \,. \end{aligned} \tag{A.16}$$

Hence, one can find the optimal portfolio shares as

$$\begin{aligned} \boldsymbol{\alpha}^{opt} &= \operatorname{argmax}\left[\mathbb{E}\left[\log\left((1+R^p)^{1-\theta}\right)\right] + \frac{1}{2}\,\mathrm{var}\left[\log\left((1+R^p)^{1-\theta}\right)\right]\right] \\ &= \operatorname{argmax}\left[\mathbb{E}(r^p) + \frac{1-\theta}{2}\mathrm{var}(r^p)\right] \end{aligned} \tag{A.17}$$

Plugging in the expression for $r^p$ one ends up with

$$\begin{aligned}
\boldsymbol{\alpha}^{opt} &= \operatorname{argmax}\left[r^f + \boldsymbol{\alpha}'\left[\mathbb{E}(\boldsymbol{r} - r^f\boldsymbol{\iota}) + \frac{\boldsymbol{\sigma^2}}{2}\right] - \frac{1}{2}\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}\,\boldsymbol{\alpha} + \frac{1-\theta}{2}\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}\,\boldsymbol{\alpha}\right] \\
&= \operatorname{argmax}\left[r^f + \boldsymbol{\alpha}'\left[\mathbb{E}(\boldsymbol{r} - r^f\boldsymbol{\iota}) + \frac{\boldsymbol{\sigma^2}}{2}\right] - \frac{\theta}{2}\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}\,\boldsymbol{\alpha}\right]. \qquad \text{(A.18)}
\end{aligned}$$

It is now easy to derive the first order condition and solve for the optimal portfolio shares,

$$0 = \mathbb{E}(\boldsymbol{r} - r^f\boldsymbol{\iota}) + \frac{\boldsymbol{\sigma^2}}{2} - \theta\boldsymbol{\Sigma}\,\boldsymbol{\alpha} \qquad \text{(A.19)}$$

$$\boldsymbol{\alpha}^{opt} = \frac{1}{\theta}\,\boldsymbol{\Sigma}^{-1}\left[\mathbb{E}(\boldsymbol{r} - r^f\boldsymbol{\iota}) + \frac{\boldsymbol{\sigma^2}}{2}\right]. \qquad \text{(A.20)}$$

## A.3 Appendix to Section 2.4

For simplicity I will present a derivation for the case with only one risky asset in addition to housing and the risk-free investment opportunity. (Pelizzon and Weber 2003) set their model in continuous time and assume that stock and house prices follow Brownian motion with zero drift and constant variance $\sigma_X$ and $\sigma_P$ respectively. The covariance between house prices and asset prices is denoted by $\sigma_{XP}$. I follow (Flavin and Nakagawa 2004) and use the amount of funds invested into stocks as the choice variable instead of the number of shares bought. To be exact, note that $\sigma_X$ really refers to the variance of the asset price and not the variance of the investment in stocks $X$. However, the equations can be read more easily using this notation. The investments' values thus evolve according to

$$\Delta P_t H_t = P_t H_t\big((\mu_H + r_f)\Delta t + \Delta\omega_{Ht} H_t\big) \tag{A.21}$$
$$\Delta X_t = X_t\big((\mu_X + r_f)\Delta t + \Delta\omega_{Xt} X_t\big) \tag{A.22}$$

where $r_f$ stands for the risk-free return, $\mu$ for each asset's expected excess return. Assume that the household does not adjust the level of the housing stock during a time interval $(0, s)$. During that interval wealth evolves according to

$$\begin{aligned}\Delta W_t = \big(r_f W_t + P_t H_0 \mu_H + X_t \mu_X - C_t\big)\Delta t \\ + X_t \Delta\omega_{Xt} + P_t H_0 \Delta\omega_{Ht}\,.\end{aligned} \tag{A.23}$$

By the Bellman principle of optimality the value function can be expresses as

$$\begin{aligned}\mathcal{V}_0(H_0, W_0) = \\ \sup_{\{C_t\}\{X_t\}} \mathbb{E}\Big\{\int_0^s e^{-\delta t} u(H_0, C_t)\Delta t + e^{-\delta t}\mathcal{V}_s(H_0, W_s)\Big\}\end{aligned} \tag{A.24}$$

subject to the budget constraint and the return processes. The integral intuitively represents the sum of utility rewards in the period $(0, s)$ plus the expected discounted value of future rewards given that behavior after period $s$ is optimal. By subtracting $\mathcal{V}(H_0, W_0)$, dividing by $s$ and letting $s$ go to zero the following expression can be found.

$$0 = \lim_{s \to 0} \sup_{\{C_t\}\{X_t\}} \mathbb{E}\Big\{ \frac{1}{s} \int_0^s e^{-\delta t} u(H_0, C_t) \Delta t + \frac{1}{s} \big( e^{-\delta t} \mathcal{V}_s(H_0, W_s) - \mathcal{V}_0(H_0, W_0) \big) \Big\} \tag{A.25}$$

By Ito's lemma, this can be rewritten as

$$0 = \sup_{\{C_t\}} \Big\{ u(H_0, C_t) - \delta \mathcal{V}_0(H_0, W_0) + \frac{\partial \mathcal{V}}{\partial W} \big( r_f W_0 + P_0 H_0 \mu_H + X_0 \mu_X - C_0 \big) \frac{\partial \mathcal{V}}{\partial P} P_0 \mu_H + \frac{1}{2} \frac{\partial^2 \mathcal{V}}{\partial W^2} \big( X_0^2 \sigma_X^2 + P_0^2 H_0^2 \sigma_P^2 + 2 P_0 H_0 X_0 \sigma_{XP} \big) \Big\} \tag{A.26}$$

$$0 \Leftrightarrow \sup_{\{C_t\}} \Big\{ u(H_0, C_t) - C_0 \frac{\partial \mathcal{V}}{\partial W} \Big\} - \delta \mathcal{V}_0(H_0, W_0) + \frac{\partial \mathcal{V}}{\partial W} \big( r_f W_0 + P_0 H_0 \mu_H \big) + \frac{1}{2} \frac{\partial^2 \mathcal{V}}{\partial W^2} P_0^2 H_0^2 \sigma_P^2 + \sup_{\{X_t\}} \Big\{ \frac{\partial \mathcal{V}}{\partial W} X_0 \mu_X + \frac{1}{2} \frac{\partial^2 \mathcal{V}}{\partial W^2} \big( X_0^2 \sigma_X^2 + 2 P_0 H_0 X_0 \sigma_{XP} \big) \Big\}. \tag{A.27}$$

The tedious derivation now leads to two first order conditions – the standard condition for non-durable consumption $\frac{\partial u}{\partial C} = \frac{\partial \mathcal{V}}{\partial W}$ and the first order condition for the investment in stocks

$$\frac{\partial \mathcal{V}}{\partial W} \mu_X + \frac{\partial^2 \mathcal{V}}{\partial W^2} \big( X_0 \sigma_X^2 + P_0 H_0 \sigma_{XP} \big) = 0 \tag{A.28}$$

which can be rearranged to yield the optimal level of investments in the risky asset and the optimal portfolio share accorded to the risky asset

$$X_0^{opt} = \left\{ \frac{-\frac{\partial \mathcal{V}}{\partial W}}{\frac{\partial^2 \mathcal{V}}{\partial W^2}} \right\} \frac{\mu_X}{\sigma_X^2} - P_0 H_0 \frac{\sigma_{XP}}{\sigma_X^2} \tag{A.29}$$

$$\alpha_X^{opt} = \left\{ \frac{-\frac{\partial \mathcal{V}}{\partial W}}{\frac{\partial^2 \mathcal{V}}{\partial W^2} W} \right\} \frac{\mu_X}{\sigma_X^2} - \alpha_H \frac{\sigma_{XP}}{\sigma_X^2} . \tag{A.30}$$

Equation (2.16) is the exact equivalent for the case with multiple risky assets and the following variance-covariance structure between returns

$$\mathbf{\Omega} = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{\Gamma}_{s\,h} \\ \mathbf{\Gamma}'_{s\,h} & \sigma_h^2 \end{pmatrix} . \tag{A.31}$$

# Bibliography


Aiyagari, S. R., and M. Gertler (1990): "Asset Returns with Transaction Costs and Uninsured Individual Risk," *Journal of Monetary Economics*, 27, 311–331.

Altig, D., A. J. Auerbach, L. J. Kotlikoff, K. A. Smetters, and J. Walliser (2001): "Simulating Fundamental Tax Reform in the United States.," *American Economic Review*, 91, 574–595.

Ando, A., and F. Modigliani (1963): "The 'Life-Cycle' Hypothesis of Saving: Aggregate Implications and Tests," *American Economic Review*, 53, 55–84.

Barsky, R. B., F. T. Juster, M. S. Kimball, and M. D. Shapiro (1997): "Preference Parameters and Behavioral Heterogeneity: An Experimental Approach in the Health and Retirement Study," *Quarterly Journal of Economics*, 112(2), 537–579.

Bertaut, C. C., and M. Starr-McCluer (2002): "Household Portfolios in the United States," in *Household Portfolios*, ed. by L. Guiso, M. Haliassos, and T. Japelli, pp. 181–218. The MIT Press.

Bodie, Z., R. C. Merton, and W. F. Samuelson (1992): "Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model," *Journal of Economic Dynamics and Control*, 16, 427–449.

Campbell, J. Y., and L. M. Viceira (2001): *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*. Oxford University Press.

——— (2002): "Appendix to Strategic Asset Allocation," on http/://kuznets.fas.harvard.edu/~campbell/papers.html.

Canner, N., N. G. Mankiw, and D. N. Weil (1997): "An Asset Allocation Puzzle," *American Economic Review*, 87(1), 181–191.

Carroll, C. D. (2000): "Why Do the Rich Save So Much?," in *Does Atlas Shrug? The Economic Consequences of Taxing the Rich*, ed. by J. B. Slemrod, pp. 463–485. Harvard University Press.

——— (2002a): "Lecture Notes on Solution Methods for Microeconomic Dynamic Stochastic Optimization Problems," Johns Hopkins University.

——— (2002b): "Portfolios of the Rich," in *Household Portfolios*, ed. by L. Guiso, M. Haliassos, and T. Japelli, pp. 389–430. The MIT Press.

Cocco, J. F. (2005): "Portfolio Choice in the Presence of Housing," *Review of Financial Studies*, forthcoming.

Cocco, J. F., F. Gomes, and P. Maenhout (2005): "Consumption and Portfolio Choice over the Life Cycle," *Journal of Financial Studies*, 18(2), 491–533.

Davis, S. J., F. Kübler, and P. Willen (2005): "Borrowing and the Demand for Equity over the Life Cycle," Discussion Paper 7, Federal Reserve Bank of Boston.

Deaton, A. (1991): "Saving and Liquidity Constraints," *Econometrica*, 59(5), 1221–1248.

Fama, E. F., and K. R. French (2002): "The Equity Premium," *Journal of Finance*, 57, 637–659.

Fernandez-Villaverde, J., and D. Krueger (2002): "Consumption and Savings over the Life Cycle: How important are Consumer Durables?," Discussion Paper 34, Stanford Institute for Economic Policy Research.

Flavin, M., and S. Nakagawa (2004): "A model of Housing in the Presence of Adjustment Costs: A Structural Interpretation of Habit Persistence," Discussion Paper 10458, NBER Working Paper.

Flavin, M., and T. Yamashita (2002): "Owner-Occupied Housing and the Composition of the Household Portfolio," *American Economic Review*, 92(1), 345–362.

Friedman, M. (1957): *A Theory of the Consumption Function.* Princeton University Press.

Gollier, C. (2002): "What does Theory Have to Say About Household Portfolios?," in *Household Portfolios*, ed. by L. Guiso, M. Haliassos, and T. Japelli, pp. 27–54. The MIT Press.

Grossman, S. J., and G. Laroque (1990): "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods," *Econometrica*, 58(1), 25–51.

Guiso, L., M. Haliassos, and T. Japelli (2002): *Household Portfolios.* The MIT Press.

Guiso, L., and T. Japelli (2002): "Household Portfolios in Italy," in *Household Portfolios*, ed. by L. Guiso, M. Haliassos, and T. Japelli, pp. 251–289. The MIT Press.

Haliassos, M., and C. C. Bertaut (1995): "Why Do So Few Hold Stocks?," *The Economic Journal*, 105, 1110–1129.

Heaton, J., and D. J. Lucas (2000): "Portfolio Choice and Asset Prices: The Importance of Entrepreneurial Risk," *Journal of Finance*, 55, 1163–1198.

Henderson, J. V., and Y. M. Ioannides (1983): "A Model of Housing Tenure Choice," *American Economic Review*, 73(1), 98–113.

Hubbard, G., J. S. Skinner, and S. Zeldes (1994): "The Importance of Precautionary Motives for Explaining Individual and Aggregate Saving," *Canegie-Rochester Conference Series on Public Policy*, 40, 59–125.

Judd, K. L. (1998): *Numerical Methods in Economics.* The MIT Press, Cambridge, Massachusetts.

King, M. A., and J. I. Leape (1987): "Asset Accumulation, Information, and the Life Cycle," Discussion Paper 2392, NBER Working Paper.

Laibson, D., A. Repetto, and J. Tobacman (2001): "A Debt Puzzle," Discussion paper, mimeo.

Markowitz, H. (1952): "Portfolio Selection," *Journal of Finance*, 7, 77–91.

——— (1992): *Mean Variance Analysis in Portfolio Choice and Capital Markets*. Basil Blackwell.

Merton, R. C. (1969): "Litetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *Review of Economics and Statistics*, 51(3), 247–257.

Modigliani, F., and R. Brumberg (1954): "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data," in *Post-Keynesian Economics*, ed. by K. K. Kurihara, pp. 388–436. Rutgers University Press, New Brunswick.

Nations, U. (2000): *World Population Prospects: The 1998 Revision*. United Nations Population Division, New York.

Paxson, C. (1990): "Borrowing Constraints and Portfolio Choice," *Quarterly Journal of Economics*, 105(2), 535–543.

Pelizzon, L., and G. Weber (2003): "Are Household Portfolios Efficient? An Analysis Conditional on Housing," Discussion Paper 3890, CEPR Discussion Paper.

Platania, J., and D. E. Schlagenhauf (2000): "Housing and Asset Holding in a Dynamic General Equilibrium Model," Discussion paper, mimeo.

Quadrini, V. (1999): "Entrepreneurship, Saving, and Social Mobility," *Journal of Economic Dynamics*, 3, 1–40.

Samuelson, P. (1970): "The Fundamental Approximation Theorem of Portfolio Analysis in terms of Means, Variances and Higher Moments," *Review of Economic Studies*, 37, 537–542.

Samuelson, P. A. (1969): "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *TheReview of Economics and Statistics*, 51(3), 239–246.

Tobin, J. (1958): "Liquidity Preference as Behavior towards Risk," *Review of Economic Studies*, 25, 68–85.

Weber, M. (1958): *The Protestant Ethic and the Spirit of Capitalism.* Charles Scribner and Sons, New York.

Yao, R., and H. H. Zhang (2005): "Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints," *Review of Financial Studies*, 18(1), 197–239.

Zeldes, S. P. (1989): "Consumption and Liquidity Constraints: An Empirical Investigation," *Journal of Political Economics*, 97(2), 305–346.