Riesz bounds of Wilson bases generated by B–Splines

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Abstract. In this short paper, we are concerned with biorthogonal Wilson bases having B–splines as well as powers of sinc functions as window functions. We prove properties of B–splines and exponential Euler splines and use these properties to estimate the Riesz bounds of the Wilson bases.

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1 Introduction

Gabor frames $\{g(x-an)e^{2\pi ibmx} : m,n \in \mathbb{Z}\}$ $(a,b \in \mathbb{R}_+)$ have found wide applications in digital signal processing, in particular in time–frequency localization of signals (cf. [11]). However, by the Balian–Low theorem, Riesz bases of the above form have necessarily bad localization properties in time or frequency. See [9, p. 108] and the references therein. Therefore Wilson [18] introduced orthonormal bases that avoid the Balian–Low phenomenon by considering functions having two peaks in frequency domain. Wilsons's suggestion was simplified to a constructive approach in [10].
A more general construction are the orthonormal local trigonometric bases proposed in [7] and [13]. Here the concept of folding operators plays a significant role (cf. [1]). In contrast to Wilson bases, local Fourier bases require the basic assumption that only immediate

neighboring windows are allowed to overlap. According to [5], we call this assumption the *two-overlapping condition.* On the other hand, local trigonometric bases can also be constructed on a nonuniform partition of the real axis.
Based on an extension of the folding concept biorthogonal local Fourier bases were examined in [5, 2]. The consideration of biorthogonal Wilson bases was addressed in [6] and for special Gaussian windows in [8].
In this paper, we are concerned with biorthogonal Wilson bases. In Section 2, we provide a simple approach to basic material concerning biorthogonal Wilson bases which differs from [6]. The approach is based on the connection of the folding concept with the Zak transform and was suggested by Bittner [3].
Based on the results in Section 2, we estimate Riesz bounds of Wilson bases with cardinal B–splines and their Fourier transforms as window functions. For this, we have to prove properties of cardinal B–splines and exponential Euler splines which may be also interesting in other contexts.

2 Biorthogonal Wilson bases

Based on the orthonormal bases $\{c_k : k \in \mathbb{N}_0\}$ and $\{s_k : k \in \mathbb{N}\}$ of $L^2([0, 1/2])$ given by

$$c_0(x) := \sqrt{2}\,,\ c_k(x) := 2\cos(2\pi kx)\,,\ s_k(x) := 2\sin(2\pi kx) \quad (k \in \mathbb{N})\,,$$

we follow [12] and introduce the functions

$$\psi_k^j(x) = \begin{cases} \sqrt{2}g(x - j/2) & k = 0, j \in \mathbb{Z} \text{ even}\,, \\ 2\,g(x - j/2)\cos(2\pi kx) & k \in \mathbb{N}, j \in \mathbb{Z} \text{ even}\,, \\ 2\,g(x - j/2)\sin(2\pi kx) & k \in \mathbb{N}, j \in \mathbb{Z} \text{ odd}\,, \end{cases} \tag{2.1}$$

where $g \in L^2(\mathbb{R})$ denotes a window function. We are interested in properties of

$$\mathcal{B}_g := \{\psi_k^{2j} : j \in \mathbb{Z}, k \in \mathbb{N}_0\} \cup \{\psi_k^{2j+1} : j \in \mathbb{Z}, k \in \mathbb{N}\}\,. \tag{2.2}$$

Clearly, a similar approach is possible with respect to other intervals than $[0, 1/2]$ and with respect to the other orthonormal bases of $L^2([0, 1/2])$ usually involved in the construction of local Fourier bases. See [1].
If $\operatorname{supp} g \subseteq [-1/4, 3/4]$, then the functions ψ_k^j satisfy a two–overlapping condition and we consider a special case of local Fourier bases.
To define a folding operator for arbitrary $g \in L^2(\mathbb{R})$ similar to the folding operator known from local Fourier bases (cf. [5, 2]), we apply the Zak transform.
The *Zak transformation* $Z : L^2(\mathbb{R}) \to L^2(\mathbb{T}^2) := L^2([0, 1]^2)$ is the unitary linear operator, which maps the orthonormal basis $\{E_{jk}(x) := e^{2\pi ijx}\mathbf{1}_{[0,1]}(x - k) : j, k \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ to the orthonormal basis $\{e_{jk}(s, t) := e^{2\pi ijs}e^{2\pi ikt} : j, k \in \mathbb{Z}\}$ of $L^2(\mathbb{T}^2)$, i.e.

$$Z(E_{jk}) = e_{jk} \quad (j, k \in \mathbb{Z}).$$

Here $\mathbf{1}_{[0,1]}$ denotes the characteristic function of $[0, 1]$. For $f \in L^2(\mathbb{R})$, the Zak transform is given by

$$Zf(s, t) = \sum_{k \in \mathbb{Z}} f(s + k)e^{2\pi ikt} \quad ((s, t) \in \mathbb{T}^2)\,. \tag{2.3}$$

Furthermore, we have

$$Zf(s+1,t) = e^{-2\pi i t} Zf(s,t)\,,\ Zf(s,t+1) = Zf(s,t)\,. \tag{2.4}$$

Let the *Fourier transform* $\hat{f} \in L^2(\mathbb{R})$ of a function $f \in L^2(\mathbb{R})$ be defined by

$$\hat{f}(v) := \int_{\mathbb{R}} f(x) e^{-2\pi i x v}\, dx.$$

Then for $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ with sufficiently fast decay of f and $\hat{f}$, e.g. $|f(x)| \le C|x|^{-1-\varepsilon}$ and $|\hat{f}(x)| \le C|x|^{-1-\varepsilon}$, the Zak transforms of f and $\hat{f}$ are related by

$$Z\hat{f}(s,t) = e^{-2\pi i s t} Zf(t,-s) \quad ((s,t) \in \mathbb{T}^2). \tag{2.5}$$

Let $I_j := [j/2, (j+1)/2]$. By (2.3) and (2.4), it is easy tho check that

$$\begin{aligned} Z(\mathbf{1}_{I_{2j}} c_k)(s,t) &= \begin{cases} c_k(s) e^{2\pi i j t} & s \in [0,1/2)\,, \\ 0 & s \in [-1/2,0)\,, \end{cases} \\ Z(\mathbf{1}_{I_{2j+1}} s_k)(s,t) &= \begin{cases} 0 & s \in [0,1/2)\,, \\ s_k(s) e^{2\pi i (j+1) t} & s \in [-1/2,0) \end{cases} \end{aligned}$$

and that

$$\begin{aligned} Z(\psi_k^{2j})(s,t) &= c_k(s) e^{2\pi i j t} Zg(s,t)\,, \\ Z(\psi_k^{2j+1})(s,t) &= -s_k(s) e^{2\pi i (j+1) t} (-Zg(s+1/2,t))\,. \end{aligned}$$

This can be rewritten as

$$\begin{pmatrix} Z\psi_k^{2j}(s,t) \\ Z\psi_k^{2j}(-s,t) \end{pmatrix} = \boldsymbol{M}_g^*(s,t) \begin{pmatrix} Z(\mathbf{1}_{I_{2j}} c_k)(s,t) \\ Z(\mathbf{1}_{I_{2j}} c_k)(-s,t) \end{pmatrix} \quad ((s,t) \in [0,1/2] \times \mathbb{T})\,, \tag{2.6}$$

$$\begin{pmatrix} Z\psi_k^{2j+1}(s,t) \\ Z\psi_k^{2j+1}(-s,t) \end{pmatrix} = \boldsymbol{M}_g^*(s,t) \begin{pmatrix} Z(\mathbf{1}_{I_{2j+1}} s_k)(s,t) \\ Z(\mathbf{1}_{I_{2j+1}} s_k)(-s,t) \end{pmatrix} \quad ((s,t) \in [0,1/2] \times \mathbb{T})\,, \tag{2.7}$$

where

$$\boldsymbol{M}_g(s,t) = \begin{pmatrix} \overline{Zg(s,t)} & \overline{Zg(-s,t)} \\ -\overline{Zg(s+1/2,t)} & \overline{Zg(-s+1/2,t)} \end{pmatrix}$$

and $\boldsymbol{M}_g^* = \bar{\boldsymbol{M}}_g^T$. This motivates the following definition of the *adjoint folding operator* $T_g^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$\begin{pmatrix} Z(T_g^* f)(s,t) \\ Z(T_g^* f)(-s,t) \end{pmatrix} = \boldsymbol{M}_g^*(s,t) \begin{pmatrix} Zf(s,t) \\ Zf(-s,t) \end{pmatrix} \quad ((s,t) \in [0,1/2] \times \mathbb{T})\,.$$

Clearly, the corresponding *folding operator* $T_g : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given by

$$\begin{pmatrix} Z(T_g f)(s,t) \\ Z(T_g f)(-s,t) \end{pmatrix} = \boldsymbol{M}_g(s,t) \begin{pmatrix} Zf(s,t) \\ Zf(-s,t) \end{pmatrix} \quad ((s,t) \in [0,1/2] \times \mathbb{T}).$$

In particular, we see by (2.6) and (2.7) that

$$Z\psi_k^{2j} = ZT_g^*(\mathbf{1}_{I_{2j}} c_k)\,,\ Z\psi_k^{2j+1} = ZT_g^*(\mathbf{1}_{I_{2j+1}} s_k). \tag{2.8}$$

In the "two–overlapping" setting, the folding operator T_g coincides with the usual folding operator for local Fourier bases on the equally partioned real axis [5, 2].
In Section 3, we examine window functions $g \in L^2(\mathbb{R})$ which are symmetric with respect to 1/4, i.e.

$$g(x) = \overline{g(1/2 - x)}\,. \tag{2.9}$$

For these window functions, we have

$$Zg(s,t) = \sum_{k\in\mathbb{Z}} g(s+k)e^{2\pi ikt} = \sum_{k\in\mathbb{Z}} \overline{g(1/2-s-k)}e^{-2\pi i(-k)t} = \overline{Zg(1/2-s,t)}$$

such that M_g has the simpler form

$$M_g(s,t) = \begin{pmatrix} \overline{Zg(s,t)} & \overline{Zg(-s,t)} \\ -Zg(-s,t) & Zg(s,t) \end{pmatrix} \quad ((s,t) \in [0,1/2] \times \mathbb{T}). \tag{2.10}$$

With the above folding concept at hand, we consider (2.2).
Remember that a set of functions $\{u_k \in L^2(\mathbb{R}) : k \in \mathbb{Z}\}$ is a *frame of* $L^2(\mathbb{R})$, if for all $f \in L^2(\mathbb{R})$ there exist constants $0 < A \leq B < \infty$ such that

$$A\,||f||^2_{L^2(\mathbb{R})} \leq \sum_{k\in\mathbb{Z}} |(f,u_k)_{L^2(\mathbb{R})}|^2 \leq B\,||f||^2_{L^2(\mathbb{R})}.$$

The best possible constants A and B are called *frame bounds.* Every function $f \in L^2(\mathbb{R})$ can be reconstructed from the values $(f,u_k)_{L^2(\mathbb{R})}$ $(k \in \mathbb{Z})$, where the convergence of

$$f = \sum_{k\in\mathbb{Z}} (f,u_k)\tilde{u}_k \tag{2.11}$$

with respect to the "most economical" $\tilde{u}_k \in L^2(\mathbb{R})$ is determined by the quotient $\frac{B-A}{B+A} = \frac{B/A-1}{B/A+1}$ which should be small (cf. [9, p. 62]). However, frame expansions of functions are in general not unique. Instead of $\{\tilde{u}_k \in L^2(\mathbb{R}) : k \in \mathbb{Z}\}$ other function systems may fulfil (2.11) too. To obtain unique representations of functions as superposition of basic functions u_k we must turn to Riesz bases.
A set of functions $\{u_k \in L^2(\mathbb{R}) : k \in \mathbb{Z}\}$ is called a *Riesz basis* of $L^2(\mathbb{R})$, if $L^2(\mathbb{R})$ is the closure of all finite linear combinations of the functions u_k $(k \in \mathbb{Z})$ and if for all $\{c_k\}_{k\in\mathbb{Z}} \in l^2$ there exist constants $0 < A \leq B < \infty$ such that

$$A\,||\{c_k\}||_{l^2} \leq ||\sum_{k\in\mathbb{Z}} c_k u_k||^2_{L^2(\mathbb{R})} \leq B\,||\{c_k\}||_{l^2}.$$

The best possible constants A and B are the *Riesz bounds.* Further, $\{u_k \in L^2(\mathbb{R}) : k \in \mathbb{Z}\}$ is an orthonormal basis if and only if $A = B = 1$. Riesz bases are precisely those that are images, under invertible bounded linear operators on $L^2(\mathbb{R})$, of orthonormal bases.

For our set $\mathcal{B}_g$, we can establish the following

Theorem 2.1. Let $g \in L^2(\mathbb{R})$. Then, for $\mathcal{B}_g$ given by (2.1) and (2.2), the following statements are equivalent:
i) $\mathcal{B}_g$ is a frame with frame bounds A, B.

ii) $\mathcal{B}_g$ is a Riesz basis with Riesz bounds A, B.

iii) There exists constants $0 < A \leq B < \infty$ such that

$$A \leq \|M_g(s,t)^{-1}\|_2^{-2}, \|M_g(s,t)\|_2^2 \leq B \quad \text{a.e. on } [0,1/2] \times \mathbb{T} \tag{2.12}$$

and A, B are the best possible constants fulfilling these inequalities. Here $\|\cdot\|_2$ denotes the spectral norm.

Furthermore, if $g \in L^2(\mathbb{R})$ satisfies the symmetry property (2.9), then (2.12) can be rewritten as

$$A \leq D_g(s,t) \leq B \quad \text{a.e. on } [0,1/2] \times \mathbb{T}, \tag{2.13}$$

where $D_g(s,t) := |Zg(s,t)|^2 + |Zg(-s,t)|^2$.

Since we are not aware of a proof of Theorem 2.1 in literature, we sketch the short proof here. For a proof of (2.13) in the case of orthonormal Wilson bases see [10]. Note further that Bittner [4] has announced more sophisticated results in this direction.

Proof. For $g \in L^2(\mathbb{R})$, with symmetry property (2.9), we have by (2.10) that

$$M_g^*(s,t)M_g(s,t) = \begin{pmatrix} D_g(s,t) & 0 \\ 0 & D_g(s,t) \end{pmatrix},$$

which yields the equivalence of (2.12) and (2.13) for these functions.

Now we show that i) $\overset{1.}{\Rightarrow}$ iii) $\overset{2.}{\Rightarrow}$ ii) $\overset{3.}{\Rightarrow}$ i).

The third implication is straightforward, since every Riesz basis is a frame.

1. By (2.8) and since Z is a unitary operator, we obtain

$$\sum_{j,k} |(f, \psi_k^j)_{L^2(\mathbb{R})}|^2 = \sum_{j,k} |(ZT_g f, Z(\mathbf{1}_{I_{2j}} c_k))_{L^2(\mathbb{R})}|^2 + \sum_{j,k} |(ZT_g f, Z(\mathbf{1}_{I_{2j+1}} s_k))_{L^2(\mathbb{R})}|^2.$$

The functions $\{\mathbf{1}_{I_{2j}} c_k\}_{j\in\mathbb{Z},k\in\mathbb{N}_0} \cup \{\mathbf{1}_{I_{2j+1}} s_k\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ form an orthonormal basis of $L^2(\mathbb{R})$. Thus, by Parseval's identity

$$\sum_{j,k} |(f, \psi_k^j)_{L^2(\mathbb{R})}|^2 = ||ZT_g f||^2_{L^2(\mathbb{T}^2)}$$

and further by definition of ZT_g

$$||ZT_g f||^2_{L^2(\mathbb{T}^2)} = \int_0^1 \int_0^{\frac{1}{2}} ||M_g(s,t) \begin{pmatrix} Zf(s,t) \\ Zf(-s,t) \end{pmatrix}||_2^2 \, ds \, dt. \tag{2.14}$$

Since on the other hand

$$\int_0^1 \int_0^{\frac{1}{2}} || \begin{pmatrix} Zf(s,t) \\ Zf(-s,t) \end{pmatrix} ||_2^2 \, ds \, dt = ||Zf||_{L^2(\mathbb{T}^2)} = ||f||_{L^2(\mathbb{R})}$$

we obtain that i) implies iii).

2. By (2.14) and since Z is a unitary operator, we see that

$$||T_g f||^2_{L^2(\mathbb{R})} = \int_0^1 \int_0^{\frac{1}{2}} ||M_g(s,t) \begin{pmatrix} Zf(s,t) \\ Zf(-s,t) \end{pmatrix} ||_2^2 \, ds \, dt \,,$$

such that T_g is a bounded linear operator with bounded inverse if M_g fulfills iii). Since we have by (2.8) that $\mathcal{B}_g = \{T_g^*(1_{I_{2j}} c_k) : j \in \mathbb{Z}, k \in \mathbb{N}_0\} \cup \{T_g^*(1_{I_{2j+1}} s_k) : j \in \mathbb{Z}, k \in \mathbb{N}\}$, this yields ii). ■

3 B–splines and their Fourier transforms as window functions

The *cardinal B–splines* N_m *of order* m are defined by

$$N_1 := \frac{1}{2}(1_{[0,1)} + 1_{(0,1]}) \,, \quad N_{m+1} := N_m * N_1 \quad (m \in \mathbb{N}) \,,$$

where $*$ denotes the convolution in $L^2(\mathbb{R})$. The *centered cardinal B–splines* M_m *of order* m are given by

$$M_m(x) := N_m(x + m/2) \,. \tag{3.1}$$

Note that $\mathrm{supp}(N_m) = [0, m]$ and that and that N_m is symmetric with respect to $m/2$, i.e. $N_m(m/2 - x) = N_m(m/2 + x)$. The Fourier transform of M_m is given by

$$\hat{M}_m(v) = (\mathrm{sinc}(v))^m \,, \tag{3.2}$$

where

$$\mathrm{sinc}(v) := \begin{cases} 1 & v = 0 \,, \\ \frac{\sin(\pi v)}{\pi v} & \text{otherwise} \,. \end{cases}$$

Moreover, B–splines fulfil the *two–scale relation*

$$N_m(x) = 2^{1-m} \sum_{k=0}^{m} \binom{m}{k} N_m(2x - k) \,. \tag{3.3}$$

We begin with the consideration of the two–overlapping case, i.e. we set $g(x) := M_m(a(x - 1/4))$ $(a \geq m)$. To determine the Riesz bounds of the corresponding Wilson bases, we have to apply the following lemma which seems to be clear at first glance.

Lemma 3.1. For $m \geq 2$, the cardinal B–splines have the following properties:
i) N'_m is monotone increasing on $[0, \frac{m+1}{4}]$,
ii) $N'_m(x) \leq N'_m\left(\frac{m}{2} - x\right)$ for all $x \in [0, \frac{m}{4}]$.

Proof. We prove the assertion by induction on m, where we mainly apply that the derivatives of cardinal B–splines fulfil (cf. [15])

$$N'_{m+1}(x) = N_m(x) - N_m(x-1) = \int_{x-1}^{x} N'_m(t)\ dt\,. \tag{3.4}$$

For the "hat function" N_2, the assertion is obvious.
Assume now that i) and ii) hold for $k \le m$.
First, we show that N'_{m+1} is monotone increasing on $[0, \frac{m+2}{4}]$.
By induction hypothesis i), we have for $t \in [0, \frac{m+1}{4}]$ that

$$N'_m(t) - N'_m(t-1) \ge 0\,.$$

Let $t \in [\frac{m+1}{4}, \frac{m+2}{4}]$ such that $t-1 \in [\frac{m-3}{4}, \frac{m-2}{4}]$ and $\frac{m}{2} - t \in [\frac{m-2}{4}, \frac{m-1}{4}]$. Then we obtain by assumption i) that

$$N'_m(t) - N'_m(t-1) \ge N'_m(t) - N'_m\left(\frac{m}{2} - t\right)$$

and further, since by induction hypothesis ii) for $t \in [\frac{m}{4}, \frac{m}{2}]$

$$N'_m\left(\frac{m}{2} - t\right) \le N'_m(t)\,,$$

that

$$N'_m(t) - N'_m(t-1) \ge 0\,.$$

Thus, we get for $0 \le x \le y \le \frac{m+2}{4}$ that

$$\begin{aligned} \int_0^x N'_m(t) - N'_m(t-1)\, dt &\le \int_0^y N'_m(t) - N'_m(t-1)\ dt\,, \\ N_m(x) - N_m(x-1) &\le N_m(y) - N_m(y-1)\,, \end{aligned}$$

which yields assertion ii) by (3.4).

Next, we prove ii). We distinguish the cases $x \in [0, \frac{1}{2}]$, $x \in [\frac{1}{2}, \frac{m-1}{4}]$ and $x \in [\frac{m-1}{4}, \frac{m+1}{4}]$.
Let $x \in [0, \frac{1}{2}]$. Then we obtain by (3.4) and since $N'_m(t) = -N'_m(m-t)$ that

$$\begin{aligned} N'_{m+1}\left(\frac{m+1}{2} - x\right) &= \int_{\frac{m+1}{2}-x-1}^{\frac{m+1}{2}-x} N'_m(t)\ dt = \int_{\frac{m-1}{2}-x}^{\frac{m}{2}} N'_m(t)\ dt + \int_{\frac{m}{2}}^{\frac{m+1}{2}-x} N'_m(t)\ dt \\ &= \int_{\frac{m}{2}-\frac{1}{2}-x}^{\frac{m}{2}-\frac{1}{2}+x} N'_m(t)\ dt \end{aligned}$$

and further by assumption ii) and i) that

$$N'_{m+1}\left(\frac{m+1}{2} - x\right) \ge \int_{-x+\frac{1}{2}}^{x+\frac{1}{2}} N'_m(t)\ dt \ge \int_0^x N'_m(t)\ dt = N'_{m+1}(x)\,.$$

Let $x \in [\frac{1}{2}, \frac{m-1}{4}]$. By (3.4) and ii), we obtain for $x \in [\frac{1}{2}, \frac{m}{4} - \frac{1}{2}]$ that

$$N'_{m+1}\left(\frac{m+1}{2} - x\right) = \int_{\frac{m-1}{2}-x}^{\frac{m+1}{2}-x} N'_m(t)\ \mathrm{d}t \geq \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} N'_m(t)\ \mathrm{d}t$$

and similarly for $x \in (\frac{m}{4} - \frac{1}{2}, \frac{m-1}{4}]$ that

$$\begin{aligned} N'_{m+1}\left(\frac{m+1}{2} - x\right) &= \int_{\frac{m-1}{2}-x}^{x+\frac{1}{2}} N'_m(t)\ \mathrm{d}t + \int_{x+\frac{1}{2}}^{\frac{m+1}{2}-x} N'_m(t)\ \mathrm{d}t \\ &\geq \int_{\frac{m-1}{2}-x}^{x+\frac{1}{2}} N'_m(t)\ \mathrm{d}t + \int_{x-\frac{1}{2}}^{\frac{m-1}{2}-x} N'_m(t)\ \mathrm{d}t = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} N'_m(t)\ \mathrm{d}t\,. \end{aligned}$$

Now assumption i) implies that

$$N'_{m+1}\left(\frac{m+1}{2} - x\right) \geq \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} N'_m(t)\ \mathrm{d}t \geq \int_{x-1}^{x} N'_m(t)\ \mathrm{d}t = N'_{m+1}(x)\,.$$

Finally, let $x \in [\frac{m-1}{4}, \frac{m+1}{4}]$. By (3.4) we obtain

$$\begin{aligned} N'_{m+1}\left(\frac{m+1}{2} - x\right) - N'_{m+1}(x) &= \int_{\frac{m-1}{2}-x}^{\frac{m+1}{2}-x} N'_m(t)\ \mathrm{d}t - \int_{x-1}^{x} N'_m(t)\ \mathrm{d}t \\ &= \int_{x}^{\frac{m+1}{2}-x} N'_m(t)\ \mathrm{d}t - \int_{x-1}^{\frac{m-1}{2}-x} N'_m(t)\ \mathrm{d}t \end{aligned}$$

$$= \int_{x}^{\frac{m+1}{4}} N'_m(t)\ \mathrm{d}t + \int_{\frac{m+1}{4}}^{\frac{m+1}{2}-x} N'_m(t)\ \mathrm{d}t - \int_{x-1}^{\frac{m-3}{4}} N'_m(t)\ \mathrm{d}t - \int_{\frac{m-3}{4}}^{\frac{m-1}{2}-x} N'_m(t)\ \mathrm{d}t\,.$$

By induction hypothesis i), we have

$$\int_{x}^{\frac{m+1}{4}} N'_m(t)\ \mathrm{d}t \geq \int_{x-1}^{\frac{m-3}{4}} N'_m(t)\ \mathrm{d}t\,,$$

while assumptions ii) and i) yield

$$\int_{\frac{m+1}{4}}^{\frac{m+1}{2}-x} N'_m(t)\ \mathrm{d}t \geq \int_{x-\frac{1}{2}}^{\frac{m-1}{4}} N'_m(t)\ \mathrm{d}t \geq \int_{\frac{m-3}{4}}^{\frac{m-1}{2}-x} N'_m(t)\ \mathrm{d}t\,.$$

Thus, we get assertion ii) for $x \in [\frac{m-1}{4}, \frac{m+1}{4}]$.
This completes the proof. ■

Theorem 3.2. Let $g(x) := M_m(a(x-1/4))$ $(m \geq 2)$. Then $\mathcal{B}_g$ is not a Riesz basis for $a \geq 2m$, while it constitutes a Riesz basis for $m \leq a < 2m$ with Riesz bounds $A_m = 2\,M_m^2(a/4)$ and $B_m = M_m^2(0)$.

Proof. Since $\mathrm{supp}(g) \subseteq [-1/4, 3/4]$ and $M_m(x) = M_m(-x)$, we obtain by (2.3) for $a \geq m$ that

$$D_g(s,t) = D_g(s) = M_m^2(as) + M_m^2(a(1/2-s)) \quad (s \in [0,1/2]).$$

We show that the function $D_g(s)$ attains its minimum on $[0,1/4]$ in $s = 1/4$ and its maximum in $s = 0$. To this end we calculate the derivative

$$D_g'(s) = 2a\left(M_m(as)M_m'(as) - M_m(a(1/2-s))M_m'(a(1/2-s))\right).$$

By Lemma 3.1, we have $M_m'(as) \leq M_m'(a(1/2-s)) \leq 0$ for $s \in [0,1/4]$.
Since further $M_m(as) \geq M_m(a(1/2-s)) \geq 0$ for $s \in [0,1/4]$, we conclude that $D_g'(s) \leq 0$ for $s \in [0,1/4]$. Consequently, we obtain by Theorem 2.1 for $m \leq a < 2m$ that $A_m = 2M_m^2(a/4) > 0$ and $B_m = 2M_m^2(0) < \infty$. For $a \geq 2m$, we see that $M_m(a/4) = 0$ such that $\mathcal{B}_g$ is not a Riesz basis. ■

To see how $C_m := B_m/A_m$ increases with m, we consider the following computation, where $a := 2m$:

m	2	6	10	22	26	30	34	38
C_{m+1}/C_m	1.778	2.2580	2.2623	2.2640	2.2641	2.2641	2.2642	2.2642

Indeed, using [17], the quotient $\lim_{m\to\infty} C_{m+1}/C_m$ can be estimated as follows: Since by Taylor expansion of the sin function

$$f_m(t) := \mathrm{sinc}\left(\sqrt{\frac{6}{m}}t\right)^m = \left(1 - \frac{\pi^2 t^2}{m} + \frac{1}{5!}\left(\frac{6}{m}\right)^2 \pi^4 t^4 - \ldots\right)^m,$$

we see that the functions f_m converge uniformely for $m \to \infty$ to $e^{-\pi^2 t^2}$. Since on the other hand by (3.2)

$$M_m(0) = \int_{-\infty}^{\infty} (\mathrm{sinc}\, v)^m \, dv = \sqrt{\frac{6}{m}} \int_{-\infty}^{\infty} \left(\mathrm{sinc}(\sqrt{\frac{6}{m}}t)\right)^m \, dt,$$

$$M_m\left(\frac{m}{4}\right) = \int_{-\infty}^{\infty} (\mathrm{sinc}\, v)^m e^{2\pi i v m/4} \, dv = \sqrt{\frac{6}{m}} \int_{-\infty}^{\infty} \left(\mathrm{sinc}(\sqrt{\frac{6}{m}}t)\right)^m e^{2\pi i\sqrt{3/(8m)}t} \, dt$$

and the Fourier transform of $e^{-x^2/b}$ is given by $\sqrt{\pi b}\, e^{-bv^2\pi^2}$, we have that

$$\lim_{m\to\infty} (M_{m+1}(0)/M_m(0))^2 = m/(m+1) = 1$$

while

$$\lim_{m\to\infty}\left(\frac{M_m(m/4)}{M_{m+1}((m+1)/4)}\right)^2 \approx \lim_{m\to\infty}\frac{m}{m+1}\,\frac{e^{-3m/4}}{e^{-3(m+1)/4}} = e^{3/4} \approx 2.117\,.$$

Preparing the next result we start with the definition of exponential Euler splines. The *exponential Euler splines* ϕ_m $(m \in \mathbb{N})$ are defined by [15]

$$\phi_m(s,t) = \sum_{k\in\mathbb{Z}} M_m(s-k)e^{2\pi ikt} \quad (s \in \mathbb{R}, t \in (-1/2, 1/2])\,. \tag{3.5}$$

The following theorem summarizes results about exponential Euler splines stated in [19].

Theorem 3.3. The exponential Euler splines ϕ_m $(m \geq 2)$ satisfy:
i) Let $s,t \in [0,1/2]$ be fixed. Then $|\phi_m(s,t)| \leq |\phi_{m-1}(s,t)|$.
ii) Let $s \in [0,1]$ be fixed. Then $|\phi_m(s,t)|$ decreases for $t \in [0,1/2]$.
Furthermore, $(s,t) = (1/2,1/2)$ is the unique root of ϕ_m on $[0,1] \times [0,1/2]$.
iii) Let $t \in [0,1/2]$ be fixed. Then $|\phi_m(s,t)|$ decreases for $s \in [0,1/2]$ and increases for $s \in [1/2,1]$.
iv) B–splines form a partition of unity, i.e. $\phi_m(s,0) = 1$ for $s \in [0,1]$.
v) The function

$$U_m(s) := \phi_m(s,1/2) = \sum_{k\in\mathbb{Z}}(-1)^k M_m(s-k)$$

decreases on $[0,1]$, where $U_m(0) > 0$ and satisfies the additional properties:

$$\begin{aligned} U_m(1-s) &= -U_m(s)\,, \\ U_m'(-s+1/2) &= U_m'(s+1/2) = -2\,U_{m-1}(s) \quad (m>2)\,, \\ U_m''(s) &= -4\,U_{m-2}(s) \quad (m>3)\,. \end{aligned}$$

Now we can formulate our next result.

Theorem 3.4. Let $g(x) := M_m(x-1/4)$ $(m \geq 2)$. Then $\mathcal{B}_g$ constitutes a Riesz basis with upper Riesz bound $B = 2$ and lower Riesz bound $A = A_m$, which can be estimated by

$$U_m^2(0)/2 \leq A_m \leq U_{m-1}^2(0)/2\,,$$

i.e. for even m by

$$2\,(1-2^{-m})^2\left(\frac{2}{\pi}\right)^{2m} \leq A_m \leq \frac{\pi^4}{8}\left(\frac{1-2^{2-m}}{1-2^{3-m}}\right)^2\left(\frac{2}{\pi}\right)^{2m},$$

and for odd m by

$$2\,(1-2^{-m-1})^2\left(\frac{2}{\pi}\right)^{2(m+1)} \leq A_m \leq \frac{\pi^4}{8}\left(\frac{1-2^{1-m}}{1-2^{2-m}}\right)^2\left(\frac{2}{\pi}\right)^{2(m+1)}.$$

Note that for sufficiently large $m \in \mathbb{N}$

$$C_{m+1}/C_m \approx A_m/A_{m+1} \approx (\pi/2)^2 \approx 2.4672.$$

Proof. By (2.3), (3.5) and since M_m is even, we obtain

$$D_g(s,t) = |\phi_m(1/4-s,t)|^2 + |\phi_m(1/4+s,t)|^2 \quad ((s,t) \in [0,1/2] \times \mathbb{T}).$$

By Theorem 3.3ii), the above function attains its minimum in $t = 1/2$ and its maximum in $t = 0$. Thus, we conclude by Theorem 2.1, that we have to look for
$A_m = \min\{D_g(s,1/2) : s \in [0,1/4]\}$ and $B_m = \max\{D_g(s,0) : s \in [0,1/4]\}$.
By Theorem 3.3iv), we see immediately that $B_m = B = 2$.
Following Theorem 3.3v), we rewrite A in the form

$$A_m = \min\{U_m^2(s) + U_m^2(1/2-s) : s \in [0,1/4]\}.$$

By straightforward computation we obtain that $A_2 = 1/2$ and $A_3 = 1/4$.
In the following, let $m > 3$. We define the linear function

$$h_m(s) := -2U_m(0)\,s + U_m(0).$$

passing through the points $(0, U_m(0))$ and $(1/2, U_m(1/2)) = (1/2, 0)$. Since we have by Theorem 3.3v) that $U_m''(s) \le 0$ for $s \in [0,1/2]$, the function U_m is concave on $[0,1/2]$. Thus, $h_m(s) \le U_m(s)$ for $s \in [0,1/2]$. On the other hand, we see by Theorem 3.3v) that

$$h_{m-1}(s) = -2\,U_{m-1}(0)s + U_{m-1}(0) = U_m'(1/2)s + U_{m-1}(0)$$

such that $U_m(s) \le h_{m-1}(s)$ for $s \in [0,1/2]$.
Now it is easy to check that $\min\{h_m^2(s) + h_m^2(1/2-s) : s \in [0,1/4]\} = U_m^2(0)/2$.
Consequently,

$$U_m^2(0)/2 \le A_m \le U_{m-1}^2(0)/2. \tag{3.6}$$

By [14], we have that

$$U_{2m}(0) = \frac{2^{2m}(2^{2m}-1)}{(2m)!}\,|B_{2m}|$$

and further since the *Bernoulli numbers* B_{2m} can be estimated by

$$\frac{2(2m)!}{(2\pi)^{2m}} < |B_{2m}| < \frac{2(2m)!}{(2\pi)^{2m}}\,\frac{2^{2m}}{2^{2m}-2}$$

that

$$\frac{2(2^{2m}-1)}{\pi^{2m}} < U_{2m}(0) < \frac{2(2^{2m}-1)}{\pi^{2m}}\,\frac{2^{2m}}{2^{2m}-2}.$$

By Theorem 3.3i), it follows $U_{2m+2}(0) \le U_{2m+1}(0) \le U_{2m}(0)$ such that

$$\frac{2(2^{2m+2}-1)}{\pi^{2m+2}} < U_{2m+1}(0) < \frac{2(2^{2m}-1)}{\pi^{2m}}\,\frac{2^{2m}}{2^{2m}-2}.$$

Together with (3.6) this yields the desired estimates for A_m. ■

Finally, we consider Wilson bases with powers of sinc–functions as window functions. Again, we prepare our result by proving some properties of B–splines.

Lemma 3.5. Let $m \geq 2$ and

$$V_m(x) := \sum_{k \in \mathbb{Z}} (-1)^k M_m(x - 2k).$$

Then, for odd $m \in \mathbb{N}$,

$$V_m(1/2) = 2^{(m-3)/2} U_m(0)$$

and for even $m \in \mathbb{N}$,

$$\begin{aligned} V_m(0) &= 2^{(m-2)/2} U_m(0), \\ 2^{(m-4)/2} U_m(0) \leq V_m(1/2) &\leq 2^{(m-2)/2} U_m(0). \end{aligned}$$

Proof. Due to the two–scale relation (3.3) we obtain

$$U_m(0) = \sum_{j \in \mathbb{Z}} (-1)^j \left(2^{1-m} \sum_{k=0}^{m} \binom{m}{k} M_m(2j + \frac{m}{2} - k) \right). \tag{3.7}$$

Let $m \in \mathbb{N}$ be odd. Then (3.7) can be rewritten as

$$U_m(0) = 2^{1-m} \sum_{l=(-m+1)/2}^{(m+1)/2} \binom{m}{\frac{m-1}{2} + l} \sum_{j \in \mathbb{Z}} (-1)^j M_m(2j + \frac{1}{2} - l).$$

Since M_m is even and

$$\binom{m}{\frac{m-1}{2} + 2r + 1} = \binom{m}{\frac{m-1}{2} - 2r},$$

we obtain by splitting the above sum into even and odd $l \in \mathbb{N}$ that

$$\begin{aligned} U_{2m}(0) &= 2^{2-m} \sum_{l=\lfloor(-m+3)/4\rfloor}^{\lfloor(m+1)/4\rfloor} \binom{m}{\frac{m-1}{2} + 2l} \sum_{j \in \mathbb{Z}} (-1)^j M_m(2j + \frac{1}{2} - 2l) \\ &= 2^{2-m} V_m\left(\frac{1}{2}\right) \sum_{l=\lfloor(-m+3)/4\rfloor}^{\lfloor(m+1)/4\rfloor} (-1)^l \binom{m}{\frac{m-1}{2} + 2l}, \end{aligned}$$

where $\lfloor x \rfloor$ denotes the integer part of x, i.e. $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. The last sum S_o has the form

$$S_o = \begin{cases} \left| \sum_{k=0}^{(m-1)/2} \binom{m}{2k} - 2 \sum_{k=0}^{(m-5)/4} \binom{m}{4k+2} \right| & m \equiv 1 \mod 8 \text{ or } m \equiv 5 \mod 8, \\ \left| \sum_{k=0}^{(m-1)/2} \binom{m}{2k+1} - 2 \sum_{k=0}^{(m-3)/4} \binom{m}{4k+3} \right| & m \equiv 3 \mod 8 \text{ or } m \equiv 7 \mod 8. \end{cases}$$

Using the formulas in [20, p. 17], we obtain that $S_u = 2^{(m-1)/2}$ and consequently $U_m(0) = 2^{(3-m)/2} V_m(1/2)$.
For the rest of the proof let $m \in \mathbb{N}$ be even. Then (3.7) can be rewritten as

$$U_m(0) = 2^{1-m} \sum_{l=-m/2}^{m/2} \binom{m}{\frac{m}{2}+l} \sum_{j \in \mathbb{Z}} (-1)^j M_m(2j-l).$$

Since M_m is even, we have for $l = 2r+1$ that

$$\sum_{j \in \mathbb{Z}} (-1)^j M_m(2j-2r-1) = (-1)^r \sum_{k \in \mathbb{Z}} (-1)^k M_m(2k-1) = 0$$

such that

$$U_m(0) = 2^{1-m} \sum_{k \in \mathbb{Z}} (-1)^k M_m(2k) \sum_{l=-\lfloor \frac{m}{4} \rfloor}^{\lfloor \frac{m}{4} \rfloor} (-1)^l \binom{m}{\frac{m}{2}+2l}.$$

The last sum S_e has the form

$$S_e = \begin{cases} \left| \sum\limits_{k=0}^{m/2} \binom{m}{2k} - 2 \sum\limits_{k=0}^{m/4} \binom{m}{4k} \right| & m \equiv 0 \mod 4, \\ \left| \sum\limits_{k=0}^{(m-2)/2} \binom{m}{2k+1} - 2 \sum\limits_{k=0}^{(m-2)/4} \binom{m}{4k+1} \right| & m \equiv 2 \mod 4. \end{cases}$$

Using [20, p. 17] again, we see that $S_e = 2^{m/2}$. Hence $U_m(0) = 2^{(2-m)/2} V_m(0)$.
To prove the last assertion we consider $V_m(x)$. Obviousely, $V_2(x) = M_2(x) = 1 - x$ for $x \in [0,1]$. Assume that $V_{m-2}(x) > 0$ for $x \in (0,1)$ and $m \geq 4$. By (3.4) and (3.1), it follows $V_m''(x) = -2V_{m-2}(x) < 0$ such that V_m is concave on $(0,1)$. Since further $V_m(0) = 2^{(m-2)/2} U_m(0) > 0$ and $V_m(1) = 0$, we obtain $V_m(x) > 0$ for $x \in (0,1)$. Now concavity of V_m yields

$$V_m(1/2) \geq \frac{1}{2}(V_m(0) + V_m(1)) = \frac{1}{2} V_m(0).$$

Using that $M_m'(x) = -M_m'(-x)$, we get $V_m'(0) = 0$. Hence, V_m has a local maximum in $x = 0$ and $V_m(1/2) \leq V_m(0)$. This completes the proof. ■

Theorem 3.6. Let $g(x) := (\mathrm{sinc}(x - 1/4))^m$ $(m \geq 2)$. Then $\mathcal{B}_g$ is a Riesz basis and the Riesz bounds $A = A_m$ and $B = B_m$ can be estimated by

$$0 < A_m \leq \begin{cases} 2^{m-1} U_m^2(0) & m \text{ odd} \\ 2^m U_m^2(0) & m \text{ even} \end{cases} \leq \begin{cases} 4 \left(\frac{2\sqrt{2}}{\pi}\right)^{2m-2} \left(\frac{1-2^{1-m}}{1-2^{2-m}}\right)^2 & m \text{ odd}, \\ 4 \left(\frac{2\sqrt{2}}{\pi}\right)^{2m} \left(\frac{1-2^{-m}}{1-2^{1-m}}\right)^2 & m \text{ even}, \end{cases}$$

$$1 + U_m^2(0) \leq B_m \leq 1 + 2U_m(0) + U_m^2(0).$$

Note that for large $m \in \mathbb{N}$

$$C_{m+1}/C_m \approx (A_m/A_{m+2})^{1/2} \approx \left(\frac{\pi}{2\sqrt{2}}\right)^2 \approx 1.2337.$$

Proof. By (2.5) and since $g = (M_m e^{2\pi i \cdot /4})\check{}$, we obtain for $((s,t) \in [0,1/2] \times \mathbb{T})$ that

$$\begin{aligned} D_g(s,t) &= |Z(M_m e^{2\pi i \cdot /4})(t,-s)|^2 + |Z(M_m e^{2\pi i \cdot /4})(t,s)|^2 \\ &= |\sum_{k\in\mathbb{Z}} M_m(t+k)e^{2\pi i k(1/4-s)}|^2 + |\sum_{k\in\mathbb{Z}} M_m(t+k)e^{2\pi i k(1/4+s)}|^2. \end{aligned}$$

By Theorem 2.1 and Theorem 3.3iii), we have to look for the minimum of D_g in $[0,1/4]\times\{1/2\}$ and for the maximum in $[0,1/4] \times \{0\}$.
Concerning the minimum we obtain by $2(|a|^2 + |b|^2) = |a+b|^2 + |a-b|^2$ that

$$\begin{aligned} D_g(s,\frac{1}{2}) &= |\sum_{k\in\mathbb{Z}} M_m(\frac{1}{2}+k)e^{2\pi i k(s-1/4)}|^2 + |\sum_{k\in\mathbb{Z}} M_m(\frac{1}{2}+k)e^{2\pi i k(s+1/4)}|^2 \\ &= 2|\sum_{k\in\mathbb{Z}} M_m(\frac{1}{2}+k)e^{2\pi i k s}\cos(\frac{\pi k}{2})|^2 + 2|\sum_{k\in\mathbb{Z}} M_m(\frac{1}{2}+k)e^{2\pi i k s}\sin(\frac{\pi k}{2})|^2 \\ &= 4|\sum_{k\in\mathbb{Z}} M_m(\frac{1}{2}+k)e^{2\pi i k s}\cos(\frac{\pi k}{2})|^2. \end{aligned}$$

For $m \le 10$ it is easy to check by straightforward computation that $D_g(s,1/2)$ has its minimum in $s=0$. However, for arbitrary $m \in \mathbb{N}$, we were not able to prove this result. Therefore $D_g(0,1/2)$ can only serve as upper bound of the minimum. Applying Lemma 3.5, we obtain

$$D_g(0,\frac{1}{2}) = 4|\sum_{k\in\mathbb{Z}}(-1)^k M_m(\frac{1}{2}+2k)|^2 \begin{cases} = 2^{m-1}\,U_m^2(0) & m \text{ odd}, \\ \le 2^m\,U_m^2(0) & m \text{ even}. \end{cases}$$

By Theorem 3.3ii), we see that $D_g(s,1/2) > 0$.
Concerning the maximum we examine

$$\begin{aligned} D_g(s,0) &= |\sum_{k\in\mathbb{Z}} M_m(k)e^{2\pi i k s}|^2 + |\sum_{k\in\mathbb{Z}} M_m(k)e^{2\pi i k(1/2-s)}|^2 \\ &= \left(M_m(0) + 2\sum_{k=1}^{\infty} M_m(k)\cos(2\pi k s)\right)^2 + \left(M_m(0) + 2\sum_{k=1}^{\infty}(-1)^k M_m(k)\cos(2\pi k s)\right)^2. \end{aligned}$$

A lower bound for the maximum of $D_g(s,0)$ is given by

$$D_g(0,0) = 1 + U_m(0)^2.$$

Regarding that $U_m(0) > 0$, an upper bound for the maximum of $D_g(s,0)$ can be obtained by $a^2 + b^2 \le (a+b)^2$, $(ab \ge 0)$, namely

$$\begin{aligned} D_g(s,0) &\le (2M_m(0) + 4\sum_{k=1}^{\infty} M_m(2k)\cos(2\pi 2ks))^2 \\ &\le 4(\sum_{k\in\mathbb{Z}} M_m(2k))^2 = (1+U_m(0))^2, \end{aligned}$$

where the last equation follows by Theorem 3.3iv) and definition of U_m. This completes the proof. ∎

Based on Section 2, biorthogonal Wilson bases with Gaussians as window functions can be examined in a different way as in [8]. For estimations of Riezs bounds and explicit constructions of dual window functions see [16].

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