Higher-Order Optimality Conditions for a Minimax

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No. 143 (1992)

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Abstract: Higher-order necessary and sufficient optimality conditions for a nonsmooth minimax problem with infinitely many constraints of inequality type are established under suitable basic assumptions and regularity conditions.

1. Introduction

We consider the minimax problem:

$$(P) \qquad \begin{aligned} &\text{minimize } \sup_{\alpha \in Q} f_\alpha(x) \\ &\text{subject to} \\ &\sup_{\beta \in B} g_\beta(x) \leq 0 \text{ and } x \in C, \end{aligned}$$

where C is a nonempty subset of a normed space X, Q and B are compact topological spaces, f_α $(\alpha \in Q)$ and g_β $(\beta \in B)$ are extended-real-valued functions on X.

First-order necessary optimality conditions for (P) without constraints of inequality type are investigated in our recent paper [7]. The results obtained there can be applied to minimax problems involving functions whose generalized directional derivatives are convex. The aim of this paper is to develop higher-order necessary and sufficient optimality conditions for (P) involving functions whose generalized directional derivatives may be nonconvex in a quite general form. Optimality conditions of this type for the case in which Q is a singleton and there is no constraint of inequality type can be found in [2], [9]. Nonsmooth analysis has produced a calculus for various directional derivatives and subgradients which yields a number of first-order necessary conditions of this type (see e.g. [10]-[12]).

The paper is organized as follows. In section 2, under a basic assumption and a regularity condition we derive higher-order necessary optimality conditions for (P), which can be applied to the lower and upper directional derivatives of order k of f_α and g_β. Section 3 is devoted to developing sufficient optimality conditions for (P) in the finite dimensional case. As in Section 2, the results obtained here can be applied for the lower and upper directional derivative of order k of f_α and order p of g_β, where k and p are positive integers.

2. Higher-Order Necessary Optimality Conditions

Denote the closure of C by $\mathrm{cl}C$. For $x_0 \in \mathrm{cl}C$ we recall (see e.g. [3]) that the contingent cone to C at x_0 is the set

$$K_C(x_0) := \{d \in X \mid \exists d_n \to d,\ \exists t_n \downarrow 0 \text{ such that } x_0 + t_n d_n \in C\}.$$

Let k be a positive integer and for each $\alpha \in Q$, $\beta \in B$, $\varphi_\alpha^{(k)}$ and $\psi_\beta^{(k)}$ be extended-real-valued functions on X. Let us introduce the following

Assumption 2.1:

(a) $\varphi_\alpha^{(k)}(0) = \psi_\beta^{(k)}(0) = 0$ *for all* $\alpha \in Q$ *and* $\beta \in B$;

(b) The mappings $\alpha \mapsto \varphi_\alpha^{(k)}(d)$ *and* $\beta \mapsto \psi_\beta^{(k)}(d)$ *are continuous for all* $d \in K_C(x_0)$;

(c) $\widehat{m} := \sup_{\alpha\in Q} f_\alpha(x_0)$ *is finite; the mappings* $\alpha \mapsto f_\alpha(x_0)$ *and* $\beta \mapsto g_\beta(x_0)$ *are upper semicontinuous (u.s.c.);*

(d) If $h_n \to 0$ *as* $n \to \infty$, *then*

$$\liminf_{n\to\infty} \left[\varphi_\alpha^{(k)}(h_n) - \varphi_\alpha^{(k)}(0)\right] \leq 0$$

uniformly in α, *and*

$$\liminf_{n\to\infty} \left[\psi_\beta^{(k)}(h_n) - \psi_\beta^{(k)}(0)\right] \leq 0$$

uniformly in β;

(e) *The mapping* $d \mapsto \sup_{\alpha \in Q_0} \varphi_\alpha^{(k)}(d)$ *is u.s.c., where* $Q_0 := \{\alpha \in Q \mid f_\alpha(x_0) = \sup_{\beta \in Q} f_\beta(x_0)\}$.

Let us introduce relations between f_α and $\varphi_\alpha^{(k)}$, g_β and $\psi_\beta^{(k)}$.

Basic Assumption 2.2: *For all* $d \in K_C(x_0)$ *and sequences* $d_n \to d$, $t_n \downarrow 0$ *satisfying* $x_0 + t_n d_n \in C$,

$$\varphi_\alpha^{(k)}(d) \geq \liminf_{n \to \infty} \frac{1}{t_n^k} [f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)]$$

uniformly in α, *and*

$$\psi_\beta^{(k)}(d) \geq \liminf_{n \to \infty} \frac{1}{t_n^k} [g_\beta(x_0 + t_n d_n) - g_\beta(x_0)]$$

uniformly in β.

Example: Recall that the lower Dini directional derivative of f_α at x_0 in the direction d is defined by (see e.g. [5], [6])

$$\underline{f_\alpha}^{(1)}(x_0; d) := \liminf_{h \to d,\ t \downarrow 0} \frac{f_\alpha(x_0 + th) - f_\alpha(x_0)}{t},$$

which is also called the contingent derivative (see [1]).

Denote $d^k := (d, \cdots, d) \in X^k$. The lower directional derivative of order k of f_α at x_0 in the direction d is defined as follows (see e.g. [9])

$$\underline{f_\alpha}^{(k)}(x_0; d^k) := k! \liminf_{h \to d,\ t \downarrow 0} \frac{1}{t^k} [f_\alpha(x_0 + th) - f_\alpha(x_0) - \sum_{j=1}^{k-1} \frac{t^j \underline{f_\alpha}^{(j)}(x_0; h^j)}{j!}].$$

Note that the mapping $d \mapsto \underline{f_\alpha}^{(k)}(x_0; d^k)$ is lower semicontinuous.

In particular, if f_α is $(k-1)$ times Fréchet differentiable on X $(k > 1)$ and the derivative of order k of f_α at x_0, $f_\alpha^{(k)}(x_0)$, exists, then $f_\alpha^{(k)}(x_0)d^k = \underline{f_\alpha}^{(k)}(x_0; d^k)'(\forall d \in X)$ (see e.g. [8]). If we take $C = X$, then $K_C(x_0) = X$. Choosing $\varphi_\alpha^{(k)}(d) := \frac{f_\alpha^{(k)}(x_0)d^k}{k!}$ we assume that $\varphi_\alpha^{(i)}(x_0) \leq 0$ $(i = 1, 2, \ldots, k-1)$ and the following limit

$$f_\alpha^{(k)}(x_0)d^k = k! \lim_{h\to d,\ t\downarrow 0} \frac{1}{t^k}\left[f_\alpha(x_0+th) - f_\alpha(x_0) - \sum_{j=1}^{k-1} \frac{t^j f_\alpha^{(j)}(x_0)h^j}{j!}\right]$$

is uniform in α. Then f_α $(\alpha \in Q)$ satisfy the basic assumption 2.2.

Let us introduce a regularity condition of the type used in [4].

Regularity Condition 2.3:

(i) For any closed sets V and W satisfying $Q_0 \subset V \subset Q$ and $B_0 \subset W \subset B$ it holds that $C_{(V,W)} := \{d \in K_C(x_0) \mid \varphi_\alpha^{(k)}(d) < 0,\ \psi_\beta^{(k)}(d) < 0\ \forall \alpha \in V,\ \forall \beta \in W\} \neq \emptyset$ implies $0 \in \mathrm{cl}C_{(V,W)}$, where $B_0 := \{\beta \in B \mid g_\beta(x_0) = 0\}$;

(ii) $\{d \in K_C(x_0) \mid \psi_\beta^{(k)}(d) \leq 0\ \forall \beta \in B_0\} \subset \mathrm{cl}\{d \in K_C(x_0) \mid \psi_\beta^{(k)}(d) < 0\ \forall \beta \in B_0\}$

We are now in a position to formulate a general necessary optimality condition of order k for (P).

Theorem 2.4:

Let x_0 be a local minimizer for (P). Assume that Assumptions 2.1.(a)-2.1.(e), the basic assumption 2.2 and the regularity condition 2.3 hold. Then

$$\sup_{\alpha\in Q_0} \varphi_\alpha^{(k)}(d) \geq 0 \text{ for all } d \in K_C(x_0) \text{ satisfying } \sup_{\beta\in B_0} \psi_\beta^{(k)}(d) \leq 0. \tag{2.1}$$

Proof: We first prove that $\sup_{\alpha\in Q_0} \varphi_\alpha^{(k)}(d) \geq 0$ for all $d \in K_C(x_0)$ satisfying $\sup_{\beta\in B_0} \psi_\beta^{(k)}(d) < 0$.

Suppose this is not true. So there is $\overline{d} \in K_C(x_0)$ satisfying $\sup_{\beta\in B_0} \psi_\beta^{(k)}(\overline{d}) < 0$ and $\sup_{\alpha\in Q_0} \varphi_\alpha^{(k)}(\overline{d}) < 0$. So for some $\mu_1 > 0$ and $\mu_2 > 0$ we have that for all $\alpha \in Q_0$, $\beta \in B_0$,

$$\varphi_\alpha^{(k)}(\overline{d}) \leq -\mu_1, \tag{2.2}$$

$$\psi_\beta^{(k)}(\overline{d}) \leq -\mu_2. \tag{2.3}$$

Define $\chi_\alpha(d) := f_\alpha(x_0) + \varphi_\alpha^{(k)}(d)$, $\zeta_\beta(d) := g_\beta(x_0) + \psi_\beta^{(k)}(d)$. By virtue of (2.2) and (2.3) we have

$$\chi_\alpha(\overline{d}) \leq \widehat{m} - \mu_1 \ (\forall \alpha \in Q_0),$$

$$\zeta_\beta(\overline{d}) \leq \mu_2 \ (\forall \beta \in B_0).$$

We now prove that there is $\widehat{d} \in K_C(x_0)$ such that

$$\chi_\alpha(\widehat{d}) < \widehat{m} \ (\forall \alpha \in Q), \tag{2.4}$$

$$\zeta_\beta(\widehat{d}) < 0 \ (\forall \beta \in B). \tag{2.5}$$

Define

$$U_1 := \{\alpha \in Q \mid \varphi_\alpha^{(k)}(\overline{d}) < -\frac{\mu_1}{2}\},$$
$$U_2 := \{\beta \in B \mid \psi_\beta^{(k)}(\overline{d}) < -\frac{\mu_2}{2}\}.$$

In view of (2.2) and (2.3) one has $Q_0 \subset U_1$ and $B_0 \subset U_2$. Moreover, $Q \setminus U_1$ and $B \setminus U_2$ are compact, as the mappings $\alpha \mapsto \varphi_\alpha^{(k)}(\overline{d})$ and $\beta \mapsto \psi_\beta^{(k)}(\overline{d})$ are continuous, and Q and B are compact. By virtue of Assumptions 2.1.(a)-2.1.(c) we can find constants $l_1 > 0$, $l_2 > 0$ such that

$$\chi_\alpha(0) = f_\alpha(x_0) + \varphi_\alpha^{(k)}(0) \leq \widehat{m} - l_1 \ (\forall \alpha \in Q \setminus U_1), \tag{2.6}$$

$$\zeta_\beta(0) = g_\beta(x_0) + \psi_\beta^{(k)}(0) \leq -l_2 \ (\forall \beta \in B \setminus U_2). \tag{2.7}$$

Since the mappings $\alpha \mapsto \varphi_\alpha^{(k)}(\overline{d})$ and $\beta \mapsto \psi_\beta^{(k)}(\overline{d})$ are continuous, it follows that

$$\varphi_\alpha^{(k)}(\overline{d}) \leq -\frac{\mu_1}{2} < 0 \ (\forall \alpha \in \mathrm{cl}U_1),$$
$$\psi_\beta^{(k)}(\overline{d}) \leq -\frac{\mu_2}{2} < 0 \ (\forall \beta \in \mathrm{cl}U_2),$$

which implies that

$$C_{(\mathrm{cl}U_1, \mathrm{cl}U_2)} = \{d \in K_C(x_0) \mid \varphi_\alpha^{(k)}(d) < 0,\ \psi_\beta^{(k)}(d) < 0 \ \forall \alpha \in \mathrm{cl}U_1,\ \forall \beta \in \mathrm{cl}U_2\}$$
$$\neq \emptyset.$$

By the regularity condition 2.3 (i), $0 \in \mathrm{cl}C_{(\mathrm{cl}U_1,\mathrm{cl}U_2)}$. This implies that there exists a sequence $h_n \to 0$ such that

$$h_n \in K_C(x_0),\ \varphi_\alpha^{(k)}(h_n) < 0\ (\forall \alpha \in U_1),\ \text{and}\ \psi_\beta^{(k)}(h_n) < 0\ (\forall \beta \in U_2),$$

whence

$$\chi_\alpha(h_n) < \widehat{m}\ (\forall \alpha \in U_1), \tag{2.8}$$

$$\zeta_\beta(h_n) < 0\ (\forall \beta \in U_2). \tag{2.9}$$

According to Assumption 2.1.(d), for $\varepsilon > 0$ there exists a subsequence $\{n_p\}$ (not depending on α) of the set $\{1, 2, \ldots\}$ such that

$$\varphi_\alpha^{(k)}(h_{n_p}) < \varphi_\alpha^{(k)}(0) + \varepsilon. \tag{2.10}$$

Combining (2.6) and (2.10) yields that for all $\alpha \in Q \setminus U_1$,

$$\begin{aligned} \chi_\alpha(h_{n_p}) &< \chi_\alpha(0) + \varepsilon \\ &\le \widehat{m} - l_1 + \varepsilon. \end{aligned}$$

For $\varepsilon < l_1$ the latter implies

$$\chi_\alpha(h_{n_p}) < \widehat{m}\ (\forall \alpha \in Q \setminus U_1). \tag{2.11}$$

On the other hand, by Assumption 2.1.(d) for $\varepsilon > 0$ there exists a subsequence $\{n_{p_s}\}$ (not depending on β) of the set $\{n_p\}$ (for convenience we write $\{n_s\}$ instead of $\{n_{p_s}\}$) such that

$$\psi_\beta^{(k)}(h_{n_s}) < \psi_\beta^{(k)}(0) + \varepsilon. \tag{2.12}$$

It follows from (2.7) and (2.12) that for all $\beta \in B \setminus U_2$,

$$\begin{aligned} \zeta_\beta(h_{n_s}) &< \zeta_\beta(0) + \varepsilon \\ &\le -l_2 + \varepsilon. \end{aligned}$$

So for $\varepsilon < l_2$,

$$\zeta_\beta(h_{n_s}) < 0 \ (\forall \beta \in B \setminus U_2),$$

which together with (2.8), (2.9) and (2.11) gives (2.4) and (2.5) for $\widehat{d} := h_{n_s}$. Hence for some $\widehat{\mu}_1 > 0$ and $\widehat{\mu}_2 > 0$ we have

$$f_\alpha(x_0) + \varphi_\alpha^{(k)}(\widehat{d}) \leq \widehat{m} - \widehat{\mu}_1 \ (\forall \alpha \in Q), \tag{2.13}$$

$$g_\beta(x_0) + \psi_\beta^{(k)}(\widehat{d}) \leq -\widehat{\mu}_2 \ (\forall \beta \in B). \tag{2.14}$$

Since $\widehat{d} \in K_C(x_0)$, there exist sequences $d_n \to \widehat{d}$, $t_n \downarrow 0$ such that $x_0 + t_n d_n \in C$. Making use of the basic assumption 2.2 we get

$$\liminf_{n\to\infty} \frac{1}{t_n^k}\left[g_\beta(x_0 + t_n d_n) - g_\beta(x_0)\right] \leq \psi_\beta^{(k)}(\widehat{d}). \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$\liminf_{n\to\infty} \frac{1}{t_n^k}\left[g_\beta(x_0 + t_n d_n) - g_\beta(x_0)\right] \leq -\widehat{\mu}_2 - g_\beta(x_0).$$

Therefore, for $\varepsilon > 0$ there is a subsequence $\{n_r\}$ (not depending on β) of the set $\{1, 2, \ldots\}$ such that

$$\frac{1}{t_{n_r}^k}\left[g_\beta(x_0 + t_{n_r} d_{n_r}) - g_\beta(x_0)\right] < -\widehat{\mu}_2 - g_\beta(x_0) + \varepsilon,$$

which implies that

$$g_\beta(x_0 + t_{n_r} d_{n_r}) < g_\beta(x_0) + t_{n_r}^k(-g_\beta(x_0)) - t_{n_r}^k(\widehat{\mu}_2 - \varepsilon).$$

Since $t_{n_r} \downarrow 0$, we can assume $t_{n_r} \leq 1$. So observing that $g_\beta(x_0) \leq 0$ we get

$$g_\beta(x_0 + t_{n_r} d_{n_r}) < -t_{n_r}^k(\widehat{\mu}_2 - \varepsilon).$$

The latter implies that $x_0 + t_{n_r} d_{n_r}$ is a feasible point of (P) for $\varepsilon < \widehat{\mu}_2$.

Next, taking account of the basic assumption 2.2, it follows from (2.13) that

$$\liminf_{r\to\infty} \frac{1}{t_{n_r}^k}\left[f_\alpha(x_0 + t_{n_r} d_{n_r}) - f_\alpha(x_0)\right] \leq \widehat{m} - \widehat{\mu}_1 - f_\alpha(x_0).$$

Hence for $\varepsilon > 0$ there is a subsequence $\{n_{r_p}\}$ (not depending on α) of the set $\{n_r\}$ (for brevity we write $\{n_p\}$ instead of $\{n_{r_p}\}$) such that

$$\frac{1}{t_{n_p}^k}\left[f_\alpha(x_0 + t_{n_p} d_{n_p}) - f_\alpha(x_0)\right] \leq \widehat{m} - \widehat{\mu}_1 - f_\alpha(x_0) + \varepsilon,$$

whence

$$f_\alpha(x_0 + t_{n_p} d_{n_p}) - f_\alpha(x_0) < t_{n_p}^k(\widehat{m} - f_\alpha(x_0)) - t_{n_p}^k(\widehat{\mu}_1 - \varepsilon).$$

Consequently, observing that $\widehat{m} - f_\alpha(x_0) \geq 0$ we get

$$f_\alpha(x_0 + t_{n_p} d_{n_p}) < \widehat{m} - t_{n_p}^k(\widehat{\mu}_1 - \varepsilon).$$

Taking $\varepsilon < \widehat{\mu} := \min\{\mu_1, \mu_2\}$, we obtain

$$\sup_{\alpha \in Q} f_\alpha(x_0 + t_{n_p} d_{n_p}) < \widehat{m},$$

which conflicts with the hypothesis that x_0 is a local minimizer for (P). So we have proved that

$$\sup_{\alpha \in Q_0} \varphi_\alpha^{(k)}(d) \geq 0 \ \forall d \in \{d \in K_C(x_0) \mid \psi_\beta^{(k)}(d) < 0 \ \forall \beta \in B_0\}.$$

By Assumption 2.1.(e), the mapping $d \mapsto \sup_{\alpha \in Q_0} \varphi_\alpha^{(k)}(d)$ is u.s.c. Hence

$$\sup_{\alpha \in Q_0} \varphi_\alpha^{(k)}(d) \geq 0 \ \forall d \in \mathrm{cl}\{d \in K_C(x_0) \mid \psi_\beta^{(k)}(d) < 0 \ \forall \beta \in B_0\}.$$

By the regularity condition 2.3 (ii) we get

$$\sup_{\alpha \in Q_0} \varphi_\alpha^{(k)}(d) \geq 0 \ \forall d \in \{d \in K_C(x_0) \mid \psi_\beta^{(k)}(d) \leq 0 \ \forall \beta \in B_0\}.$$

The proof is complete. $\square$

Theorem 2.4 can be applied to the lower directional derivative of order k of f_α and g_β. We obtain the following

Corollary 2.5:

Assume that for $\varphi_\alpha^{(k)}(d) := \frac{1}{k!}\underline{f_\alpha}^{(k)}(x_0; d^k)$ *and* $\psi_\beta^{(k)}(d) := \frac{1}{k!}\underline{g_\beta}^{(k)}(x_0; d^k)$ *all the hypotheses of Theorem 2.4 are satisfied. Then*

$$\sup_{\alpha \in Q_0} \underline{f_\alpha}^{(k)}(x_0; d^k) \geq 0 \quad \textit{for all } d \in K_C(x_0)$$

$$\textit{satisfying } \sup_{\beta \in B_0} \underline{g_\beta}^{(k)}(x_0; d^k) \leq 0.$$

We note that Theorem 2.4 remains true, if the basic assumption 2.2 is replaced by the stronger

Assumption 2.2': *For all* $d \in K_C(x_0)$ *and sequences* $d_n \to d$, $t_n \downarrow 0$ *satisfying* $x_0 + t_n d_n \in C$,

$$\varphi_\alpha^{(k)}(d) \geq \limsup_{n \to \infty} \frac{1}{t_n^k} [f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)]$$

uniformly in α, *and*

$$\psi_\beta^{(k)}(d) \geq \limsup_{n \to \infty} \frac{1}{t_n^k} [g_\beta(x_0 + t_n d_n) - g_\beta(x_0)]$$

uniformly in β.

Assumption 2.2' is more naturally suited for working with upper directional derivatives

Recall that the upper Dini directional derivative of f_α at x_0 in the direction d is defined as follows (see e.g. [5], [6]).

$$\overline{f}_\alpha^{(1)}(x_0; d) := \limsup_{h \to d,\ t \downarrow 0} \frac{f_\alpha(x_0 + th) - f_\alpha(x_0)}{t};$$

the upper directional derivative of order k of f_α at x_0 in the direction d is defined as follows (see e.g. [9])

$$\overline{f}_\alpha^{(k)}(x_0; d^k) := k! \limsup_{h \to d,\ t \downarrow 0} \frac{1}{t^k} \left[f_\alpha(x_0 + th) - f_\alpha(x_0) - \sum_{j=1}^{k-1} \frac{t^j \overline{f}_\alpha^{(j)}(x_0; h^j)}{j!} \right].$$

Applying Theorem 2.4 to upper directional derivative we obtain

Corollary 2.6:

Assume that for $\varphi_\alpha^{(k)}(d) := \frac{1}{k!}\overline{f}_\alpha(x_0; d^k)$ *and* $\psi_\beta^{(k)}(d) := \frac{1}{k!}\overline{g}_\beta^{(k)}(x_0; d^k)$ *all the hypotheses of Theorem 2.4 are fulfilled, where the basic assumptions 2.2 is replaced by Assumption 2.2'. Then*

$$\sup_{\alpha\in Q_0} \overline{f}_\alpha^{(k)}(x_0; d^k) \geq 0 \quad \textit{for all } d \in K_C(x_0)$$

$$\textit{satisfying } \sup_{\beta\in B_0} \overline{g}_\beta^{(k)}(x_0; d^k) \leq 0.$$

3. Higher-Order Sufficient Optimality Conditions

In this section we deal with the case in which $X = I\!R^m$.

Definition 3.1 ([12]): The point $x_0 \in C$ is said to be a strict local minimizer of order k for the mathematical program $\min\{f(x) \mid x \in C\}$ if there exist a number $\sigma > 0$ and a neighborhood U of x_0 such that

$$f(x) > f(x_0) + \sigma\|x - x_0\|^k \text{ for all } x \in U \cap C,\ x \neq x_0.$$

Let x_0 be a feasible point of (P). Let k and p be positive integers and let $\varphi_\alpha^{(i)}$ $(\alpha \in Q)$ and $\psi_\beta^{(j)}$ $(\beta \in B)$ be extended-real-valued functions on $I\!R^m$ $(i = 1,\ldots,k;\ j = 1,\ldots,p)$. We define

$$M(x_0) := \{d \in K_C(x_0) \mid \varphi_\alpha^{(i)}(d) \leq 0\ \forall\alpha \in Q_0,\ i = 1.\ldots,k-1;$$
$$\psi_\beta^{(j)}(d) \leq 0\ \forall\beta \in B_0,\ j = 1,\ldots,p\}.$$

Let us introduce relations between f_α and $\varphi_\alpha^{(i)}$, g_β and $\psi_\beta^{(j)}$ $(i = 1,\ldots,k;\ j = 1,\ldots,p)$.

Basic Assumption 3.2: *For all* $d \in K_C(x_0)$ *and sequences* $d_n \to d,\ t_n \downarrow 0$ *satisfying* $x_0 + t_n d_n \in C$, *and for each* $i = 1,\ldots,k$ *and* $j = 1,\ldots,p)$ *there hold*

$$\varphi_\alpha^{(i)}(d) \leq \limsup_{n\to\infty} \frac{1}{t_n^i}\left[f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)\right],$$

$$\psi_\beta^{(j)}(d) \leq \limsup_{n\to\infty} \frac{1}{t_n^j}\left[g_\beta(x_0+t_nd_n)-g_\beta(x_0)\right].$$

A higher-order sufficient optimality condition for (P) can be stated as follows.

Theorem 3.3:

Let $\sup_{\alpha\in Q} f_\alpha(x_0)$ be finite and the basic assumption 3.2 hold. Assume that

$$\sup_{\alpha\in Q_0} \varphi_\alpha^{(k)}(d) > 0 \text{ for all } d \in M(x_0)\setminus\{0\}. \tag{3.1}$$

Then x_0 is a strict local minimizer of order k for (P).

Proof: Assume that x_0 is not a strict local minimizer of order k for (P). Then for any $\sigma > 0$ there exists a sequence $x_n \to x_0$, $x_n \in C$, $x_n \neq x_0$ such that

$$\sup_{\alpha\in Q} f_\alpha(x_n) \leq \sup_{\alpha\in Q} f_\alpha(x_0) + \sigma\|x_n-x_0\|^k, \tag{3.2}$$

$$g_\beta(x_n) \leq 0 \ (\forall\beta\in B). \tag{3.3}$$

By virtue of (3.2) in particular for $\alpha \in Q_0$, we get

$$f_\alpha(x_n) \leq f_\alpha(x_0) + \sigma\|x_n-x_0\|^k. \tag{3.4}$$

Taking $t_n = \|x_n - x_0\|$ and $d_n = \frac{x_n-x_0}{\|x_n-x_0\|}$ we obtain $x_0 + t_nd_n = x_n \in C$. Moreover, since the set $\{d \in I\!R^m \mid \|d\| = 1\}$ is compact, we may assume that $d_n \to d_0$ with $\|d_0\| = 1$. Hence $d_0 \in K_C(x_0)$.

Furthermore, for each $\alpha \in Q_0$, $i = 1,\ldots,k-1$ it follows from the basic assumption 3.2 and (3.4) that

$$\begin{aligned}\varphi_\alpha^{(i)} &\leq \limsup_{n\to\infty} \frac{f_\alpha(x_0+t_nd_n)-f_\alpha(x_0)}{t_n^i}\\ &\leq \lim_{n\to\infty} \sigma t_n^{k-i} = 0.\end{aligned}$$

On the other hand, from (3.3) it follows that

$$g_\beta(x_0+t_nd_n) - g_\beta(x_0) \leq 0 \ (\forall\beta\in B_0). \tag{3.5}$$

According to the basic assumption 3.2 we have

$$\psi_\beta^{(j)}(d_0) \leq \limsup_{n\to\infty} \frac{1}{t_n^k}\left[g_\beta(x_0+t_nd_n) - g_\beta(x_0)\right]. \tag{3.6}$$

Combining (3.5) and (3.6) yields that

$$\psi_\beta^{(j)}(d_0) \leq 0 \ (\forall \beta \in B_0;\ j = 1,\ldots,p).$$

Consequently, $d_0 \in M(x_0) \setminus \{0\}$.

Making use of (3.4) we get

$$\frac{f_\alpha(x_0+t_nd_n) - f_\alpha(x_0)}{t_n^k} \leq \sigma \ (\forall \alpha \in Q_0).$$

So taking account of the basic assumption 3.2 yields that there is a subsequence $\{n_s\}$ of the set $\{1,2,\ldots\}$ such that

$$\varphi_\alpha^{(k)}(d_0) - \frac{\sigma}{2} < \frac{f_\alpha(x_0+t_{n_s}d_{n_s}) - f_\alpha(x_0)}{t_{n_s}^k} \leq \sigma,$$

which implies that

$$\varphi_\alpha^{(k)}(d_0) < \frac{3}{2}\sigma \ (\forall \alpha \in Q_0).$$

Since σ is arbitrary, this implies

$$\varphi_\alpha^{(k)}(d_0) \leq 0 \ (\forall \alpha \in Q_0),$$

which contradicts (3.1). This completes the proof. □

Theorem 3.3 can be applied to the upper directional derivatives of order k of f_α and order p of g_β.

Corollary 3.4:

Assume that for $\varphi_\alpha^{(i)}(d) := \frac{1}{i!}\overline{f}_\alpha^{(i)}(x_0;d^i)$ and $\psi_\beta^{(j)}(d) := \frac{1}{j!}\overline{g}_\beta^{(j)}(x_0;d^j)$ ($i = 1,\ldots,k;\ j = 1,\ldots,p$) all the hypotheses of Theorem 3.3 are fulfilled. Then x_0 is a strict local minimizer of order k for (P).

Remarks: (α) Theorem 3.3 remains valid without the sets Q and B being compact.

(β) Theorem 3.3 remains true, if we replace "limsup" in the basic assumption 3.2 by "liminf". Then it can be applied to the lower directional derivatives of order k of f_α and of order p of g_β.

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