Einstein Equation and Geometric Quantization

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1 Introduction

Let (X, g) be a Lorentz manifold. In geometric quantization the quantum operator $\hat{H}$ of the function $H(x) := g(x,x), x \in T^*X$, is given by

$$\hat{H}(\psi) = \hbar^2(-\Box^g(\psi) + 1/6R^g\psi), \tag{1}$$

where ψ is a squared integrable function on X with respect to the density $\sqrt{-\det(g)}$, $\Box^g$ is the d'Alembert operator and R^g the scalar curvature of the metric g. Because of the fact, that the critical points (Lorentz metrics) of the functional $\int_X R^g\sqrt{-\det(g)}\,d^4x$ are determined by the Einstein equation for the vacuum, the question arises, as to whether there is a connection between the Einstein equation and the operator $\hat{H}$. In fact, we will show, that under some restrictions on the function ψ, the critical metrics of the expectation value of $\hat{H}$ have to satisfy Einstein's equation for a suitable energy-momentum tensor. Moreover, if g is a solution of the Einstein equation, the function ψ obeys the Klein-Gordon equation

$$-\Box^g(\psi) + \frac{1}{6}R^g\psi = 0.$$

These observations yield an interpretation for the scalar curvature as a density of the expectation value of the mass-squared operator $\hat{H}$ (cf.3).

2 Geometric Preliminaries

In this section we present some terminology of geometric quantization on a Lorentz manifold needed in the sequel. For more informations see [3]. Let (X, g) be a Lorentz

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manifold and (T^*X, ω_0) the cotangent bundle with the canonical symplectic structure given by the exterior derivative of the canonical one-form θ_0 on T^*X. The vertical polarization P^V on the symplectic manifold T^*X determines the Hilbert space $\mathcal{H}^{P^V}$ to be $L^2(X, g)$. The quantum operator $\hat{H}$ of the function $H : T^*X \longrightarrow \mathbb{R}$ defined by

$$H(x) = g(x, x), \quad x \in T^*X$$

is given by

$$\hat{H}(\psi) = \hbar^2(-\Box^g(\psi) + 1/6R^g\psi), \quad \psi \in D(\hat{H}),$$

where $\Box^g$ denotes the d'Alembert operator, R^g the scalar curvature of the metric g and $D(\hat{H})$ the domain of the operator $\hat{H}$. In the next section we study the expectation value of this operator.

3 The expectation value of the mass-squared operator

The *expectation value* of an operator A on $L^2(X, g)$ in the state ψ is given by

$$\langle A\rangle^\psi = \int_X A(\psi)\,\overline{\psi}\,\sqrt{-\det(g)}\,d^4x. \tag{2}$$

Associated with a Lorentz metric g and a real and nowhere vanishing state $\psi \in D(\hat{H})$, we define the Lorentz metric $\bar{g}$ by

$$\bar{g} := \psi^2\, g.$$

The expectation value of the operator $\hat{H}$ is connected to the scalar curvarure $R^{\bar{g}}$ of the metric $\bar{g}$ as follows:

Theorem 3.1

$$\frac{1}{6}\int_X R^{\bar{g}}\sqrt{-\det(\bar{g})}\,d^4x = \frac{1}{\hbar^2}\langle\hat{H}\rangle^\psi, \tag{3}$$

saying, that the Einstein-Hilbert action determined by the metric $\bar{g}$ is equal to the expectation value of the mass-squared operator $\hat{H}$. Thus $R^{\bar{g}}$ can be interpreted as a $\bar{g}$-density of $\langle\hat{H}\rangle^\psi$.

The proof follows immediately from the definition of the expectation value and the following Lemma:

Lemma 3.2 Let ψ be real and nowhere vanishing. Then the scalar curvature $R^{\bar{g}}$ of the metric $\bar{g} = \psi^2\, g$ satisfies

$$R^{\bar{g}}\psi^3 = -6\Box^g(\psi) + R^g\psi. \tag{4}$$

The Ricci tensor $Ric^{\bar{g}}$ is given by

$$Ric^{\bar{g}} = Ric^g + 4\frac{d\psi\otimes d\psi}{\psi^2} - 2\frac{\nabla^g d\psi}{\psi} - \left(\frac{|d\psi|^2}{\psi^2} + \frac{\Box^g(\psi)}{\psi}\right) g. \tag{5}$$

These formulas are a consequence of the following formula for the Ricci tensor of the conformally modified metric

$$\tilde{g} = \exp(2\,f)\,g,$$

where f is an arbitrary real, smooth function. According to [1]

$$Ric^{\tilde{g}} = Ric^{g} - 2(\nabla^{g}df - df \otimes df) + (-\Box^{g}(f) - 2|df|^2)g, \tag{6}$$

where ∇^g denotes the Levi-Civita connection of g. Inserting

$$f := \log(\psi)$$

into (6) yields (5). Taking the trace with respect to the metric $\bar{g}$ of (5) we obtain (4).

Next, we are looking at the critical points (Lorentz metrics) of the variational derivative of the expectation value $\langle \hat{H} \rangle^{\psi}$. It is well known, that the critical metrics of the Einstein-Hilbert action

$$S_{\psi^2 g} = \frac{1}{6}\int_X R^{\bar{g}} \sqrt{-\det(\bar{g})}\, d^4x$$

are entirely determined by

$$Ric^{\bar{g}} - \frac{1}{2}R^{\bar{g}}\bar{g} = 0.$$

Lemma 3.2 shows that this equation is equivalent to

$$Ric^{g} - \frac{1}{2}R^{g}g = -4\frac{d\psi \otimes d\psi}{\psi^2} + 2\frac{\nabla^{g}d\psi}{\psi} + \left(\frac{|d\psi|^2}{\psi^2} - 2\frac{\Box^{g}(\psi)}{\psi}\right) g. \tag{7}$$

Taking the trace with respect to g yields immediately the Klein-Gordon equation

$$-\Box^{g}(\psi) + \frac{1}{6}R^{g}\psi = 0 \tag{8}$$

for the state ψ. In summarizing we state:

Theorem 3.3 *Let (X, g) be a Lorentz manifold and $\psi \in D(\hat{H})$ positiv und real. A metrik $\bar{g} := \psi^2 g$ is a stationary point of the action*

$$\frac{1}{6}\int_X R^{\bar{g}} \sqrt{-\det(\bar{g})}\, d^4x,$$

iff the Einstein equation

$$Ric^{g} - \frac{1}{2}R^{g}g = -4\frac{d\psi \otimes d\psi}{\psi^2} + 2\frac{\nabla^{g}d\psi}{\psi} + \left(\frac{|d\psi|^2}{\psi^2} - 2\frac{\Box^{g}(\psi)}{\psi}\right) \tag{9}$$

is valid. Moreover, every solution g of the Einstein equation (9) requires ψ to solve the Klein-Gordon equation (8).

The expectation value of the mass-squared operator $\hat{H}$ depends on the metric g and the state ψ. Therefore it is of interest to look at the critical points of the variational derivative if either ψ or g is fixed. First let g be fixed and $\bar{R}_{ij}$ denote the components of the Ricci tensor $Ric^{\bar{g}}$ of the metric $\bar{g}$. One easily verifies

$$\begin{aligned}
&\frac{1}{6}\delta^{\psi}\int_X R^{\bar{g}}\sqrt{\det(-\bar{g})}\,d^4x \\
=\ &\frac{1}{6}\int_X(-2\bar{g}^{ij}\,\overline{R}_{ij}\,\psi^3 + 4R^{\bar{g}}\,\psi^3)\,\delta\psi\,\sqrt{\det(-g)}\,d^4x \\
=\ &\frac{1}{6}\int_X 2R^{\bar{g}}\,\psi^3\,\delta\psi\,\sqrt{\det(-g)}\,d^4x \\
=\ &\frac{1}{6}\int_X(-12\Box^g(\psi) + 2R^g\psi)\,\delta\psi\,\sqrt{\det(-g)}\,d^4x.
\end{aligned}$$

The last equation follows again from (4). Thus

$$\frac{1}{6}\delta^{\psi}\int_X R^{\bar{g}}\,\sqrt{\det(-\bar{g})}\,d^4x = 0 \Longleftrightarrow -\Box^g(\psi) + \frac{1}{6}R^g\psi = 0, \tag{10}$$

which means, that the Euler-Lagrange equation of the action $1/6\int_X R^{\bar{g}}\,\sqrt{\det(-\bar{g})}\,d^4x$ is the Klein-Gordon equation. If we don't change ψ and vary g only, we obtain

$$\begin{aligned}
&\delta^g\int_X R^{\bar{g}}\sqrt{\det(\bar{g})}\,d^4x \\
=\ &\int_X(\overline{R}_{ij}\,\delta\bar{g}^{ij}\sqrt{\det(\bar{g})} - 1/2\,\bar{g}_{ij}R^{\bar{g}}\psi^2\sqrt{\det(g)}\,\delta g^{ij})\,d^4x \\
=\ &\int_X(\overline{R}_{ij} - \frac{R^{\bar{g}}}{2}\bar{g}_{ij})\psi^2\,\delta g^{ij}\,\sqrt{\det(g)}\,d^4x.
\end{aligned} \tag{11}$$

Thus

$$\delta^g\int_X R^{\bar{g}}\,\sqrt{\det(-\bar{g})}\,d^4x = 0 \Longleftrightarrow Ric^{\bar{g}} - \frac{R^{\bar{g}}}{2}\bar{g} = 0.$$

But this is exactly the Einstein equation (9). Because of Theorem 3.1 we get the following:

Theorem 3.4 *Let ψ be a smooth, real and positive state in the domain of $\hat{H}$. The expectation value of the operator $\hat{H}$ is extremal for a fixed metric g, iff ψ fullfils the Klein-Gordon equation (8). On the other hand, if g is varied, the critical Lorentz metrics of $\langle\hat{H}\rangle^{\psi}$ in a fixed state ψ satisfy Einstein's equation (9) and ψ is a solution of the Klein-Gordon equation (8).*

We point out, that our concept generalizes to complex-valued states in an obvious manner. The interested reader is refered to [2].

Literatur

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