Bivariate Interpolation by Splines and Approximation Order

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Abstract. We construct Hermite interpolation sets for bivariate spline spaces of arbitrary degree and smoothness one on non-rectangular domains with uniform type triangulations. This is done by applying a general method for constructing Lagrange interpolation sets for bivariate spline spaces of arbitrary degree and smoothness. It is shown that Hermite interpolation yields (nearly) optimal approximation order. Applications to data fitting problems and numerical examples are given.

§1 Introduction

Tensor products of univariate splines are used for interpolation on rectangular domains. We investigate interpolation on non-rectangular domains and use bivariate spline functions. We consider spaces $S_q^r(\Delta^\mu)$, $\mu = 1, 2$, of bivariate splines of degree q and smoothness r on uniform type partitions (see §2). (For simplicity, the splines are defined on rectangular domains but the results in this paper analogously hold for simply connected non-rectangular domains as in Figure 6.)

In [11] (see also [10]), we developed algorithms for constructing point sets which admit unique Lagrange interpolation from $S_q^r(\Delta^\mu)$, $\mu = 1, 2$. By using these methods, we will construct Hermite interpolation sets for $S_q^1(\Delta^\mu)$, $\mu = 1, 2$. It is shown that Hermite interpolation yields (nearly) optimal approximation order for $S_q^1(\Delta^1)$, $q \geq 4$, and $S_q^1(\Delta^2)$, $q \geq 2$. By using Bernstein-Bézier techniques, (nearly) optimal approximation order of interpolation was proved for the following spaces: Sha [14] and Chui & He [3] for $S_2^1(\Delta^2)$, Sha [15] for $S_3^1(\Delta^1)$; and Jeeawock-Zedek [6] for $S_3^1(\Delta^2)$ (approximation order two).

In practice we use Lagrange configurations which are "close" to Hermite configurations. The interpolating splines can be computed by passing from one triangle to the next, where only small systems have to be solved. Our methods can also be applied to data fitting problems.

§2 Dimension, basis and interpolation by S_q^r

We consider bivariate spline spaces of the following type. First, the space of *bivariate polynomials of total degree q* is denoted by $\widetilde{\Pi}_q = \text{span}\{x^i y^j : i \geq 0, j \geq 0, i+j \leq q\}$. Let a rectangle $T = [a,b] \times [c,d]$ and points $a = x_0 < x_1 < \cdots < x_{n_1} = b$, $c = y_0 < y_1 < \cdots < y_{n_2} = d$ such that $x_i - x_{i-1} = h_1$, $i = 1, \ldots, n_1$; $y_j - y_{j-1} = h_2$, $j = 1, \ldots, n_2$, be given. We set $h = \max\{h_1, h_2\}$. By defining $R_{i,j} = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$, $i = 1, \ldots, n_1$; $j = 1, \ldots, n_2$, we obtain a partition of T into subrectangles $R_{i,j}$. If the diagonal from (x_{i-1}, y_{j-1}) to (x_i, y_j) is added to each subrectangle $R_{i,j}$, then we denote the resulting partition by Δ^1. If we add both diagonals to each subrectangle, then the resulting partition is denoted by Δ^2.

The spline spaces are defined as follows. Let integers r and q with $0 \leq r < q$ be given. For $\mu = 1, 2$, the space $S_q^r(\Delta^\mu)$ of all functions $f \in C^r(T)$ such that the restriction to each subset of the partition Δ^μ is in $\widetilde{\Pi}_q$ is called space of *bivariate splines of degree q and smoothness r* with respect to the partition Δ^μ. We sometimes denote these spaces simply by S_q^r.

The dimension of spline spaces for uniform partitions was determined by Chui & Wang [4] and Schumaker [13]. We begin with a result on the basis of such spaces which is due to Chui & Wang [4] and Dahmen & Micchelli [5].

Theorem 1. *A basis of S_q^r is given by the following functions:*

(i) The polynomials $x^i y^j$, $i + j \leq q$.

(ii) The truncated power functions $(a_\mu + b_\mu x + c_\mu y)_+^{r+1} x^i y^j$, $i+j \leq q-r-1$, where $a_\mu + b_\mu x + c_\mu y = 0$ are the interior grid lines of the partition.

(iii) Certain cone splines, easily defined by univariate B-splines (see below).

Roughly speaking, the cone splines in (iii) are defined as follows. Let an interior grid point z (say $z = (0,0)$) of the partition be given and denote by C the cone formed by the three respectively four lines of the partition with vertex z. A cone spline is a spline in S_q^r which vanishes on the complement of C. We choose a line l which crosses C. For $n = q, q-1, \ldots$, let B be a univariate B-spline on l in S_n^r (if it exists) and extend B continuously to be (up to a constant) a univariate truncated power function t_+^n on each ray in C. All bivariate splines obtained in this way are exactly the cone spline in (iii). (For details see [8].)

We now investigate interpolation by S_q^r. In contrast to the univariate case, it is a non-trivial problem to construct any set at which interpolation by S_q^r is possible. Therefore, we formulate the following problem: Determine a set $\{z_1, \ldots, z_N\}$ in T, where $N = \dim S_q^r$, such that for each function $f \in C(T)$, the *Lagrange interpolation problem* $s(z_i) = f(z_i)$, $i = 1, \ldots, N$, has a unique solution $s \in S_q^r$. Such a set $\{z_1, \ldots, z_N\}$ is called *Lagrange* *interpolation set* for S_q^r. If we consider not only the function values of f but also partial derivatives of f, then we speak of a *Hermite interpolation problem* for the space S_q^r, and the corresponding sets are called *Hermite interpolation sets* for S_q^r.

Given a point $z = (x, y) \in T$, we set $D^i f(z) = (f_{x^i}(z), f_{x^{i-1}y}(z), \ldots, f_{xy^{i-1}}(z), f_{y^i}(z))$. The uniform norm of f is defined by $\|f\| = \max_{z \in T} |f(z)|$ and for the derivatives, we set $\|D^i f\| = \max\{\|f_{x^\alpha y^\beta}\| : \alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta = i\}$.

In the next sections, we construct Lagrange interpolation sets for $S_q^1(\Delta^\mu)$, $\mu = 1, 2$, and then obtain Hermite interpolation sets by "taking limits". The construction of Lagrange interpolation sets is a special case of the algorithms for $S_q^r(\Delta^\mu)$, $\mu = 1, 2$, developed by Nürnberger & Riessinger [11] (see Adam [1] for arbitrary cross-cut partitions). There, the basic principle is to reduce the bivariate spline interpolation problem to a series of univariate problems and to apply simultaneously a principle of degree reduction.

In the following, we briefly describe the method for $S_q^r = S_q^r(\Delta^1)$ by means of Figure 1. In the description, we will not be very precise but try to show the basic ideas. We construct points $z_1, \ldots, z_N \in T$, where $N = \dim S_q^r$, and show that $s(z_i) = 0$, $i = 1, \ldots, N$, implies $s = 0$ on T for all $s \in S_q^r$. In the first step, we choose $q + 1$ lines $l_1, \ldots, l_{q+1}$ through T_1 and choose $\dim S_{q+1-j}^r|_{l_j \cap T_1}$ points on $l_j \cap T_1$, $j = 1, \ldots, q + 1$. It is well known that $s = 0$ on T_1 if s vanishes on the chosen points. Then we choose $q - r$ lines, again denoted by $l_1, \ldots, l_{q-r}$, through T_2 and choose $\dim\{s \in S_{q+1-j}^r : s = 0 \text{ on } T_1\}|_{l_j \cap (T_1 \cup T_2)}$ points on $l_j \cap T_2$, $j = 1, \ldots, q - r$. Then it follows (see [11]) that $s = 0$ on $T_1 \cup T_2$ if s vanishes on the chosen points. By repeating the last step, it follows that $s = 0$ on $\bigcup_{i=1}^5 T_i$ if s vanishes on the chosen points. Finally, we choose "diagonal" lines, again denoted by $l_1, l_2, \ldots$ through T_4, T_5, T_6, T_7 and choose $\dim\{s \in S_{q+1-j}^r : s = 0 \text{ on } \bigcup_{i=1}^5 T_i\}|_{l_j}$ points on $l_j \cap (T_6 \cup T_7)$, satisfying the Schoenberg-Whitney condition, $j = 1, 2, \ldots$ (until the above dimension is zero). By choosing "diagonal" lines, we do not get the grid points as interpolation points. Therefore, as in Figure 1, we choose "broken lines" through $T_6 \cup T_7$ and leave the chosen point constellation unchanged. Then it follows (see [11]) that $s = 0$ on $\bigcup_{i=1}^7 T_i$ if s vanishes on the chosen points. We continue this method until we get $s = 0$ on T. Therefore, the constructed set $\{z_1, \ldots, z_N\}$ is an interpolation set. In particular, the dimension of S_q^r is the sum of the dimension of univariate spline spaces.

§3 Interpolation by S_q^1 and approximation order

We first construct Lagrange interpolation sets for $S_q^1(\Delta^1)$, $q \geq 4$, by applying the algorithms of Nürnberger & Riessinger [11]. By "taking limits", we then obtain Hermite interpolation sets and show that interpolation at these sets yields (nearly) optimal approximation order.

For constructing Lagrange interpolation sets for $S_q^1(\Delta^1)$, $q \geq 4$, we only have to describe four basic steps. For an arbitrary subtriangle V of the partition Δ^1, one of the following steps will be applied to V.

Step A. (Starting step) Choose $q+1$ disjoint line segments $a_1, \ldots, a_{q+1}$ in V. For $i = 1, \ldots, q+1$, choose $q+2-i$ distinct points on a_i.

Step B. Choose $q-1$ disjoint line segments $b_1, \ldots, b_{q-1}$ in V. For $i = 1, \ldots, q-1$, choose $q-i$ distinct points on b_i.

Step C. Choose $q-2$ disjoint line segments $c_1, \ldots, c_{q-2}$ in V. For $i = 1, \ldots, q-2$, choose $q-i$ distinct points on c_i.

Step D. Choose $q-3$ disjoint line segments $d_1, \ldots, d_{q-3}$ in V. For $i = 1, \ldots, q-3$, choose $q-i-2$ distinct points on d_i.

Given a partition Δ^1, we apply the above steps to the subtriangles of Δ^1 as indicated in Figure 2, where we choose horizontal, vertical and diagonal line segments as indicated in Figure 1.

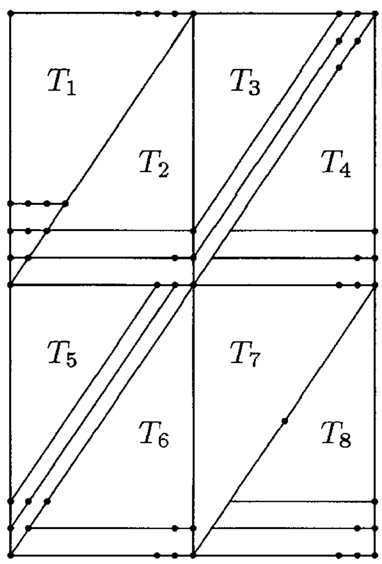

Figure 1.

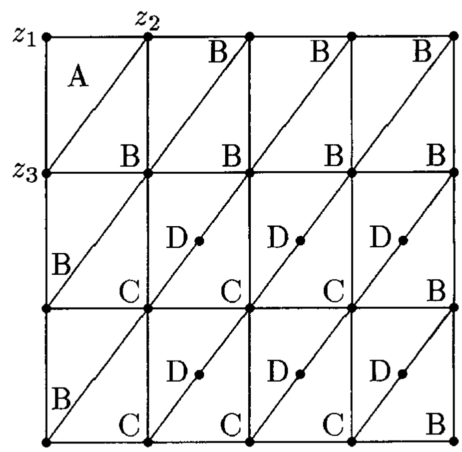

Figure 2.

The following construction of Hermite interpolation sets for $S_q^1(\Delta^1)$, $q \geq 4$, and Theorem 2 is given in Nürnberger [9].

Let a sufficiently differentiable function $f \in C(T)$ be given. For defining Hermite interpolation conditions for a spline $s \in S_q^1(\Delta^1)$, $q \geq 4$, we only have to describe four basic conditions. Let V be an arbitrary subtriangle of the partition Δ^1. We impose one of the following conditions on the polynomial $p = s|_V \in \widetilde{\Pi}_q$, where z is a vertex respectively a midpoint of an edge as indicated in Figure 2.

Condition A. (Starting condition) Any set of conditions which admits unique Hermite interpolation by $\widetilde{\Pi}_q$ (*e.g.*, $p(z_1) = f(z_1)$, $p_{x^i}(z_2) = f_{x^i}(z_2)$, $i = 0, \ldots, q-1$, $D^i p(z_3) = D^i f(z_3)$, $i = 0, \ldots, q-1$, see Figure 2).

Condition B. $D^i p(z) = D^i f(z)$, $i = 0, \ldots, q-2$.

Condition C. $D^i p(z) = D^i f(z)$, $i = 0, \ldots, q-2$, except $p_{y^{q-2}}(z) = f_{y^{q-2}}(z)$.

Condition D. $D^i p(z) = D^i f(z)$, $i = 0, \ldots, q-4$.

Given a partition Δ^1, the distribution of the Hermite interpolation conditions to the subtriangles is the same as for Lagrange interpolation and is indicated in Figure 2.

In Theorem 2, the norm denotes the maximum of the uniform norm over all subtriangles of the partition (w.r.t. the polynomial pieces).

Theorem 2. *For each function $f \in C^{q+1}(T)$, there exists a constant $K > 0$ such that for the unique spline $s \in S_q^1(\Delta^1)$ which satisfies the above Hermite interpolation conditions, the following statements hold: For all $i \in \{0, \ldots, \rho - 1\}$, $\|D^i(f - s)\| \leq K h^{\rho - i}$, where $\rho = 4$ if $q = 4$, and $\rho = q + 1$ if $q \geq 5$. (The constant $K > 0$ depends on $\|D^{q+1} f\|$ and is independent of h.)*

In the following, we construct Lagrange interpolation sets for $S_q^1(\Delta^2)$, $q \geq 2$, by applying the algorithm of Nürnberger & Riessinger [11]. By "taking limits", we then obtain Hermite interpolation sets and show that interpolation at these sets yields (nearly) optimal approximation order.

For constructing Lagrange interpolation sets for $S_q^1(\Delta^2)$, $q \geq 2$, we only have to describe four basic steps. For an arbitrary subtriangle V of the partition Δ^2, one of the following steps will be applied to V.

Step A. (Starting step) Choose $q+1$ disjoint line segments $a_1, \ldots, a_{q+1}$ in V. For $i = 1, \ldots, q+1$, choose $q + 2 - i$ distinct points on a_i.

Step B. Choose $q - 1$ disjoint line segments $b_1, \ldots, b_{q-1}$ in V. For $i = 1, \ldots, q-1$, choose $q - i$ distinct points on b_i.

Step C. Choose $q - 3$ disjoint line segments $c_1, \ldots, c_{q-3}$ in V. For $i = 1, \ldots, q-3$, choose $q - 2 - i$ distinct points on c_i.

Step D. Choose $q - 2$ disjoint line segments $d_1, \ldots, d_{q-2}$ in V. For $i = 1, \ldots, q-2$, choose $q - i$ distinct points on d_i.

Given a partition Δ^2, we apply the above steps to the subtriangles of Δ^2 as indicated in Figure 4, where we choose horizontal, vertical and diagonal line segments as indicated in Figure 3.

The following construction of Hermite interpolation sets for $S_q^1(\Delta^2)$, $q \geq 2$, and Theorem 3 will be given in Nürnberger & Walz [12].

Let a sufficiently differentiable function $f \in C(T)$ be given. For defining Hermite interpolation conditions for a spline $s \in S_q^1(\Delta^2)$, $q \geq 2$, we only

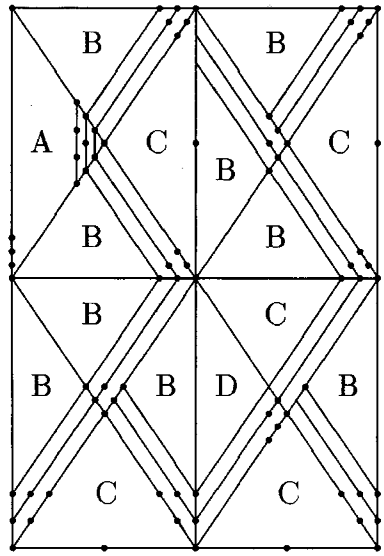

Figure 3.

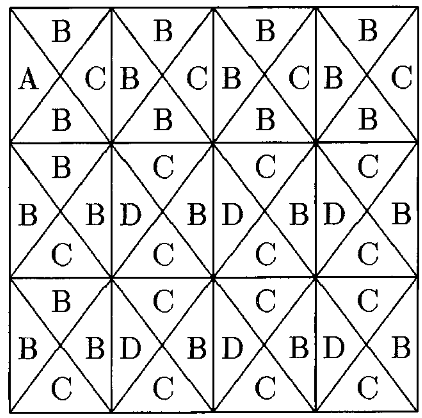

Figure 4.

have to describe four basic steps. Let V be an arbitrary subtriangle of the partition Δ^2. We impose one of the following conditions on the polynomial $p = s|_V \in \widetilde{\Pi}_q$, where z is a vertex respectively a midpoint of an edge as indicated in Figure 5.

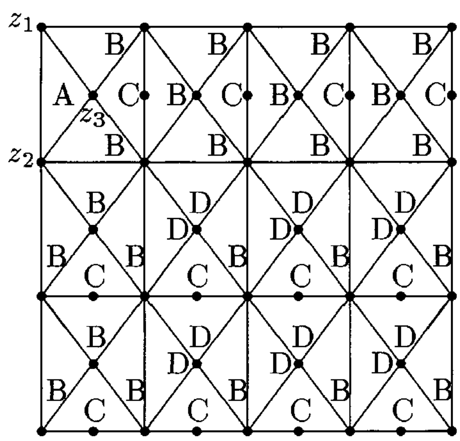

Figure 5.

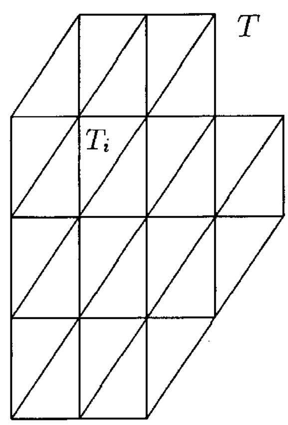

Figure 6.

Condition A. (Starting condition) Any set of conditions which admits unique Hermite interpolation by $\widetilde{\Pi}_q$ (*e.g.*, $p(z_1) = f(z_1)$, $p_{y^i}(z_2) = f_{y^i}(z_2)$, $i = 0, \ldots, q-1$, $D^i p(z_3) = D^i f(z_3)$, $i = 0, \ldots, q-1$, see Figure 5).

Condition B. $D^i p(z) = D^i f(z)$, $i = 0, \ldots, q-2$.

Condition C. $D^i p(z) = D^i f(z)$, $i = 0, \ldots, q-4$.

Condition D. $p_{r^i \bar{r}^j}(z) = f_{r^i \bar{r}^j}(z)$, $i \geq 0$, $j \geq 0$, $i+j \leq q-2$, $j \neq q-2$ (where $r = (r_1, r_2)$ is the unit vector in direction of the diagonal of the partition Δ^1 and $\bar{r} = (-r_1, r_2)$).

Given a partition Δ^2, we assign the above conditions to the subtriangles of Δ^2 as indicated in Figure 5. The distribution of the Hermite conditions is different from the distribution of the Lagrange conditions (cf. Figure 4). To see this, let two adjacent subtriangles in the upper half of a subrectangle with the Lagrange conditions D and C be given. Then by "taking limits", we get the Hermite condition D for both triangles. This follows from the fact that for all $i \in \{0,1\}$ and $j \in \{0,\dots,q\}$, $p_{r^i\bar{r}^j}(z) = \widetilde{p}_{r^i\bar{r}^j}(z)$, where $p \in \widetilde{\Pi}_q$ and $\widetilde{p} \in \widetilde{\Pi}_q$ are the two adjacent polynomial pieces.

Theorem 3. *For each function $f \in C^{q+1}(T)$, there exists a constant $K > 0$ such that for the unique spline $s \in S_q^1(\Delta^2)$ which satisfies the above Hermite interpolation conditions, the following statements hold: For all $i \in \{0,\dots,\rho-1\}$, $\|D^i(f - s_f)\| \le Kh^{\rho-i}$, where $\rho = q$ if $q \in \{2,3\}$, and $\rho = q+1$ if $q \ge 4$. (The constant $K > 0$ depends on $\|D^{q+1}f\|$ and is independent of h).*

By using Bernstein-Bézier techniques, similar results on interpolation by $S_2^1(\Delta^2)$ were proved by Sha [14] and Chui & He [3]. Moreover, Jeeawock-Zedek [6] proved approximation order 2 for interpolation by $S_3^1(\Delta^2)$ by using the same Hermite interpolation conditions as above for this space.

We briefly discuss the application of our spline interpolation methods to *data fitting*. Let a non-rectangular domain T as in Figure 6, points $w_i \in T$ and corresponding data f_i be given which we want to approximate by $S_q^1(\Delta^\mu)$.

In the first step, we interpolate the data f_i by a piecewise polynomial $\widetilde{s}$ of degree q such that $\|f - \widetilde{s}\| = O(h^{q+1})$ if $f_i = f(w_i)$ and $f \in C^{q+1}(T)$. In general, piecewise polynomial interpolation is a simpler problem than spline interpolation and in any case, this is always possible if the data are regularly distributed over T (cf. Chui [2]). In the second step, we interpolate the resulting function $\widetilde{s}$ (which may not even be continuous) by a differentiable spline $s \in S_q^1$ as described in this paper. As in Theorems 2 and 3, it can be shown that $|f_i - s(w_i)| = O(h^q)$ or $O(h^{q+1})$ (see [9] and [12]).

Finally, we give a numerical example. As a test, we interpolate Franke's test function $f(x,y) = \frac{3}{4}\,\mathrm{e}^{-\frac{(9x-2)^2+(9y-2)^2}{4}} + \frac{3}{4}\,\mathrm{e}^{-\frac{(9x+1)^2}{49}-\frac{9y+1}{10}} + \frac{1}{2}\,\mathrm{e}^{-\frac{(9x-7)^2+(9y-3)^2}{4}} - \frac{1}{5}\,\mathrm{e}^{-(9x-4)^2-(9y-7)^2}$ by splines in $S_4^1(\Delta^1)$ on $T = [0,1] \times [0,1]$ and approximate regularly distributed data which come from f (as described above). By using 1731 interpolation points, we obtain the errors $4.02 \cdot 10^{-4}$ for interpolation and $2.41 \cdot 10^{-4}$ for data fitting.

Most of the methods and results in this paper (except Theorem 1) analogously hold when the partitions Δ^1 or Δ^2 are non-uniform (see [9] and [12]).

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