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Some Results about the Large Deviations Principle in White Noise Analysis

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1. Introduction

In recent years, there has been an enormous interest in the theory of large deviations, i.e. in the asymptotic behaviour of small probabilities on an exponential scale. Although the roots of this theory can be dated back to Cramér ([Cr 38]) in 1938 it took until the mid-1970's that starting with Donsker and Varadhan ([DV 75 a,b, DV 76, DV 83]) the subject exploded. Numbers of publications have been written since then (cf. e.g. the references quoted in [DS 89] or [DZ 93]), and the subject has found many applications to related fields like statistical mechanics (e.g. [El 85, El 95]) or others.

On the other hand white noise analysis provides lots of powerful tools as well for infinite-dimensional calculus as for probability theory, a quite complete overview is given by [HKP 93].

So the combination of these two subjects should inspire new results and give a feed-back to each of them.

In the present paper I will do a very first step towards this aim stating some large deviations results in the context of white noise analysis. Not only the white noise probability measure μ will be considered, but also a certain class of functionals over the white noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ turns out to correspond to measures as first shown independently in [KS 76] and [Yo 90]. For some of these measures large deviations results can be shown as well.

2. Mathematical Preliminaries

Let as usual $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ denote the white noise probability space and for $p \in \mathbb{Z}$ let $(\mathcal{S})_p$ denote the space of functionals on $\mathcal{S}'(\mathbb{R})$ having finite norm $\|\varphi\|_p := \|\Gamma(H^p)\varphi\|_2$. Here $\Gamma(H^p)$ denotes the second quantization of the pth power of the harmonic oszillator H and $\|\cdot\|_2$ the $L^2(\mathcal{S}'(\mathbb{R}))$-norm.

The space of Hida test functions $(\mathcal{S})$ is defined as the projective limit of the spaces $(\mathcal{S})_p$ and the space $(\mathcal{S})^*$ of Hida distributions as its dual.

For more details the reader shold confer [HKP 93] and the references quoted therein.

For an element $\Phi \in (\mathcal{S})^*$ we define positivity by

Definition 2.1 An element $\Phi \in (\mathcal{S})^*$ is called positive if for every μ-a.e. positive element $\varphi \in (\mathcal{S})$ $<\Phi, \varphi> \geq 0$ holds.

The cone of positive elements in $(\mathcal{S})^*$ is denoted by $(\mathcal{S})^*_+$.

The important thing about these positive functionals is that they correspond to finite measures on $(\mathcal{S}'(\mathbb{R}),\mathcal{B})$, namely the following theorem holds:

Theorem 2.2 Let $\Phi \in (\mathcal{S})^*_+$. Then there exists a finite measure ν_Φ on $(\mathcal{S}'(\mathbb{R}),\mathcal{B})$, so that for all $\varphi \in (\mathcal{S})$

$$< \Phi, \varphi > = \int\limits_{\mathcal{S}'(\mathbb{R})} \widetilde{\varphi}(x)\, d\nu_\Phi(x)$$

holds, where $\widetilde{\varphi}$ is the pointwise defined, continuous version of φ.

A proof of this theorem can be found in [HKP 93].

Since large deviations theory states results for the behaviour of measures with variance decreasing to 0, we define for a measure ν

Definition 2.3 For $\varepsilon > 0$ and ν a finite measure on $(\mathcal{S}'(\mathbb{R}),\mathcal{B})$ we define ν_ε to be the image of the measure ν under the transformation $x \to \sqrt{\varepsilon}\, x$, i.e. for $A \in \mathcal{B}$ $\nu_\varepsilon(A)$ is defined as $\nu\left(\frac{1}{\sqrt{\varepsilon}} A\right)$.

Next we give some basic definitions from the large deviations theory, where for simplicity we have already used the notations as in white noise analysis.

Definition 2.4 A function $I : \mathcal{S}'(\mathbb{R}) \longrightarrow [0,\infty]$ is called a rate function if it is lower semicontinuous, i.e. if for all $\alpha \in \mathbb{R}^+$ the set $I^{-1}([0,\alpha])$ is closed.

I is called a good rate function if $I^{-1}([0,\alpha])$ is compact.

Definition 2.5 A family of measures $\{\nu_\varepsilon\}$ on $(\mathcal{S}'(\mathbb{R}),\mathcal{B})$, defined as in definition 2.3, is said to satisfy the strong large deviations principle with the rate function I if the following holds:

$$\underline{\lim}_{\varepsilon \longrightarrow 0} \varepsilon \log \nu_\varepsilon(G) \;\geq\; - \inf_{x \in G} I(x) \qquad \forall G \subseteq \mathcal{S}'(\mathbb{R}) \text{ open}$$

and

$$\overline{\lim}_{\varepsilon \longrightarrow 0} \varepsilon \log \nu_\varepsilon(F) \;\leq\; - \inf_{x \in F} I(x) \qquad \forall F \subseteq \mathcal{S}'(\mathbb{R}) \text{ closed.}$$

Definition 2.6 A family of measures $\{\nu_\varepsilon\}$ on $(\mathcal{S}'(\mathbb{R}),\mathcal{B})$ is called exponentially tight if for every $L > 0$ there exists a compact subset $K_L \subseteq \mathcal{S}'(\mathbb{R})$ such that

$$\overline{\lim}_{\varepsilon \longrightarrow 0} \varepsilon \log \nu_\varepsilon(K_L^c) \;\leq\; -L$$

where of course K_L^c denotes the complement of K_L in $\mathcal{S}'(\mathbb{R})$.

The most important tool to proof large deviations results in white noise analysis seems to be Baldi's theorem:

Theorem 2.7 Let $\{\nu_\varepsilon\}$ be a family of measures on $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$ with

(i) $\{\nu_\varepsilon\}$ is exponentially tight.

(ii) For all $\xi \in \mathcal{S}(\mathbb{R})$

$$\Lambda(\xi) := \lim_{\varepsilon \to 0} \varepsilon \log \int_{\mathcal{S}'(\mathbb{R})} e^{<x, \frac{\xi}{\varepsilon}>} \, d\nu_\varepsilon(x)$$

exists.

(iii) Assume finally that with

$$\Lambda^*(x) := \sup_{\xi \in \mathcal{S}(\mathbb{R})} (< x, \xi > - \Lambda(\xi))$$

the Fenchel-Legendre transform of Λ and

$$E := \Big\{ x \in \mathcal{S}'(\mathbb{R}) \Big| \exists \xi \in \mathcal{S}(\mathbb{R}) \text{ with } < x, \xi > - \Lambda^*(x) \; > \; < z, \xi > - \Lambda^*(z) \; \forall z \neq x \text{ and } \Lambda(\gamma\xi) < \infty \text{ for some } \gamma > 1 \Big\}$$

the set of exposed points $\inf_{x \in G \cap E} \Lambda^*(x) = \inf_{x \in G} \Lambda^*(x)$ holds for all $G \subseteq \mathcal{S}'(\mathbb{R})$ open.

Then the family $\{\nu_\varepsilon\}$ satisfies the strong large deviations principle with the good rate function Λ^*.

A proof of this theorem can be found in [DZ 93].

3. Large Deviation Results in White Noise Analysis

The aim is now to show that the large deviations principle holds for some families of measures on $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$. Since actually we do not really work on $\mathcal{S}'(\mathbb{R})$ but on some subspace $\mathcal{S}_{-p}(\mathbb{R})$ – which is the subspace of those elements of $\mathcal{S}'(\mathbb{R})$ having finite norm $|x|_{-p} := |H^{-p}x|_2$ – first it is to show that the problem is well-defined.

Lemma 3.1 The support $\mathcal{S}_{-p}(\mathbb{R})$ of the measures $\{\nu_\varepsilon\}$ does not depend on ε.

Proof: In the Minlos theorem (cf. [Mi 62] for a detailed presentation) it is stated that if the characteristic function

$$C(\xi) := \int_{\mathcal{S}'(\mathbb{R})} e^{i<x,\xi>} \, d\nu(x)$$

is continuous with respect to some $|\cdot|_q$, $q \in \mathbb{N}$ and if $p \in \mathbb{N}$ is such that the injection from $\mathcal{S}_p(\mathbb{R})$ into $\mathcal{S}_q(\mathbb{R})$ is of Hilbert-Schmidt type, then $\mathcal{S}_{-p}(\mathbb{R})$ is the full support of the measure ν.

So all we have to show is that if the characteristic function C of ν is continuous with respect to some norm $|\cdot|_q$, then so are the characteristic functions C_ε of the measures ν_ε. But this is obvious since $C_\varepsilon(\xi) = C(\sqrt{\varepsilon}\xi)$ as an easy computation shows. $\square$

The next result I'm going to show is that for all $\Phi \in (\mathcal{S})^*_+$ the corresponding family of measures $\{\nu_\varepsilon\} := \{(\nu_\Phi)_\varepsilon\}$, defined as in theorem 2.2 and definition 2.3, is exponentially tight. First I need a lemma:

Lemma 3.2 For given $q \in I\!N$ there exists a $p \in I\!N$ such that the functional $e^{|\cdot|^2_{-p}} \in (\mathcal{S})_q$.

Proof: For $x \in \mathcal{S}_{-p}(I\!R)$ the function $\Phi(x) = e^{|x|^2_{-p}}$ has chaos decomposition

$$\Phi^{(n)}(x_1,\ldots,x_n) = \begin{cases} 1 & n=0 \\ 0 & n \text{ odd} \\ \frac{1}{m!}\widehat{\Psi^{(n)}}(x_1,\ldots,x_n) & n=2m \end{cases}$$

where $\widehat{\Psi^{(n)}}$ denotes the symmetrization of

$$\Psi^{(n)}(x_1,\ldots,x_n) = < x_1, H^{-2p}x_2 > \cdots < x_{n-1}, H^{-2p}x_n > .$$

Hence – with $\|\cdot\|_{\text{H.S.}}$ denoting the Hilbert-Schmidt norm – for given q

$$\begin{aligned} \|\Phi\|_q^2 &= \sum_n n! \left\|(H^{\otimes n})^q \Phi^{(n)}\right\|^2_{\text{H.S.}} \\ &= \sum_m (2m)! \left\|(H^{\otimes 2m})^q \frac{1}{m!}\widehat{\Psi^{(2m)}}\right\|^2_{\text{H.S.}} \\ &\leq \sum_m (2m)! \left(\frac{1}{m!}\right)^2 \left\|(H^{\otimes 2m})^q \Psi^{(2m)}\right\|^2_{\text{H.S.}} \end{aligned}$$

and for $\left\|(H^{\otimes 2m})^q \Psi^{(2m)}\right\|^2_{\text{H.S.}}$ we get the estimation

$$\begin{aligned} &\left\|(H^{\otimes 2m})^q \Psi^{(2m)}\right\|^2_{\text{H.S.}} \\ &= \sum_{e_{k_1},\ldots,e_{k_{2m}}} \left|(H^{\otimes 2m})^q < e_{k_1}, H^{-2p}e_{k_2} > \ldots < e_{k_{2m-1}}, H^{-2p}e_{k_{2m}} >\right|^2 \\ &= \sum_{k_1,\ldots,k_m=0}^{\infty} (2k_1+2)^{-4(p-q)} \cdots (2k_m+2)^{-4(p-q)} \\ &= \left[2^{-4(p-q)} \sum_{k=0}^{\infty} (k+1)^{-4(p-q)}\right]^m \\ &\leq 8^{-m} \end{aligned}$$

for p large enough. Here the e_k denote sequences of Hermite polynomials which are eigenvectors of the harmonic oszillator.

So we get for $\|\Phi\|_q^2$ and p chosen as above the estimation

$$\begin{aligned} \|\Phi\|_q^2 &\leq \sum_{m=0}^{\infty} (2m)! \left(\frac{1}{m!}\right)^2 8^{-m} \\ &\leq \sum_{m=0}^{\infty} 4^m (m!)^2 \left(\frac{1}{m!}\right)^2 8^{-m} \\ &= 2. \end{aligned}$$

□

Theorem 3.3 Let $\Phi \in (\mathcal{S})^*_+$. Then the family of corresponding measures $\{\nu_\varepsilon\} := \{(\nu_\Phi)_\varepsilon\}$ is exponentially tight.

Proof: Suppose $\Phi \in (\mathcal{S})^*_+ \cap (\mathcal{S})_{-q}$. Then choose $p \in I\!N$ such that $\mathcal{S}_{-p}(I\!R)$ is a support of ν_Φ and $e^{|\cdot|^2_{-p}} \in (\mathcal{S})_q$. For given $L > 0$ choose $K_L := \{x \in \mathcal{S}'(I\!R) : |x|^2_{-p} \le L\}$. K_L is compact in $\mathcal{S}_{-p}(I\!R)$ with the weak topology, and since the injection from $\mathcal{S}_{-p}(I\!R)$ with the weak topology into $\mathcal{S}'(I\!R)$ with the weak topology is continuous (cf. e.g. [Tr 67], 35.8) K_L is compact in $\mathcal{S}'(I\!R)$ as well.

Now we get the estimation

$$\begin{aligned}
\nu_\varepsilon(K_L^c) &= \int\limits_{|x|^2_{-p} > L} d\nu_\varepsilon(x) \\
&= \int\limits_{e^{\frac{1}{\varepsilon}|x|^2_{-p}} > e^{\frac{L}{\varepsilon}}} d\nu_\varepsilon(x) \\
&\le e^{-\frac{L}{\varepsilon}} \int\limits_{\mathcal{S}'(I\!R)} e^{\frac{1}{\varepsilon}|x|^2_{-p}}\, d\nu_\varepsilon(x) \\
&= e^{-\frac{L}{\varepsilon}} < \Phi, e^{|\cdot|^2_{-p}} >
\end{aligned}$$

where we used the Chebycheff-Markov inequality. Since $e^{|\cdot|^2_{-p}} \in (\mathcal{S})_q$ we get

$$\overline{\lim_{\varepsilon \longrightarrow 0}}\ \varepsilon \log \nu_\varepsilon(K_L^c) \le \lim_{\varepsilon \longrightarrow 0} \varepsilon \log\left(e^{-\frac{L}{\varepsilon}} < \Phi, e^{|\cdot|^2_{-p}} >\right) = -L.$$

$\square$

The next result actually is not really surprising since analogous results are known for Gaussian measures in other contexts.

Before I'm going to state it I want to quote a duality lemma for Fenchel-Legendre transforms, applied to our case:

Lemma 3.4 Let $f : \mathcal{S}'(I\!R) \longrightarrow (-\infty, \infty]$ be lower semicontinuous and convex and define

$$g(\xi) := \sup_{x \in \mathcal{S}'(I\!R)} \left(< x, \xi > - f(x)\right).$$

Then f is the Fenchel-Legendre transform of g, i.e.

$$f(x) = \sup_{\xi \in \mathcal{S}(I\!R)} \left(< x, \xi > - g(\xi)\right).$$

For a proof of this lemma in a more general case cf. e.g. [DZ 93].

Theorem 3.5 Let $(\mathcal{S}'(I\!R), \mathcal{B}, \mu)$ the white noise space. Then for the family $\{\mu_\varepsilon\}$, defined as in definition 2.3, the strong large deviation principle holds with the good rate function

$$I(x) = \begin{cases} \frac{1}{2}|x|_2^2 & x \in L^2(I\!R) \\ \infty & \text{otherwise} \end{cases}.$$

Proof: In view of Baldi's theorem there only is to show:

(i) $\{\mu_\varepsilon\}$ is exponentially tight.

(ii) For all $\xi \in \mathcal{S}(I\!R)$

$$\Lambda(\xi) \;:=\; \lim_{\varepsilon \longrightarrow 0} \varepsilon \log \int\limits_{\mathcal{S}'(I\!R)} e^{<x,\frac{\xi}{\varepsilon}>} \, d\mu_\varepsilon(x)$$

exists.

(iii) The Fenchel-Legendre transform of Λ is

$$\Lambda^*(x) \;=\; \begin{cases} \frac{1}{2}|x|_2^2 & x \in L^2(I\!R) \\ \infty & \text{otherwise} \end{cases} .$$

(iv) With E the set of exposed points defined as in theorem 2.7

$$\inf_{x \in G \cap E} \Lambda^*(x) \;=\; \inf_{x \in G} \Lambda^*(x)$$

holds for all $G \subseteq \mathcal{S}'(I\!R)$ open.

This can be shown as follows:

(i) With $\Phi(x) \equiv 1$ this is just a special case of theorem 3.3.

(ii) Since

$$\int\limits_{\mathcal{S}'(I\!R)} e^{<x,\frac{\xi}{\varepsilon}>} \, d\mu_\varepsilon(x) \;=\; e^{\frac{1}{2\varepsilon}|\xi|_2^2}$$

obviously

$$\Lambda(\xi) \;=\; \lim_{\varepsilon \longrightarrow 0} \varepsilon \log \int\limits_{\mathcal{S}'(I\!R)} e^{<x,\frac{\xi}{\varepsilon}>} \, d\mu_\varepsilon(x) \;=\; \frac{1}{2}|\xi|_2^2$$

exists for all $\xi \in \mathcal{S}(I\!R)$.

(iii) In view of lemma 3.4 there only is to show that with

$$I(x) \;=\; \begin{cases} \frac{1}{2}|x|_2^2 & x \in L^2(I\!R) \\ \infty & \text{otherwise} \end{cases}$$

$$\sup_{x \in \mathcal{S}'(I\!R)} \left(< x, \xi > - I(x) \right) \;=\; \frac{1}{2}|\xi|_2^2$$

holds. Obviously we have

$$\sup_{x \in \mathcal{S}'(I\!R)} \left(< x, \xi > - I(x) \right) \;=\; \sup_{x \in L^2(I\!R)} \left(< x, \xi > - \frac{1}{2}|x|_2^2 \right)$$

and for all $x \in L^2(I\!R)$

$$\left(< x, \xi > - \frac{1}{2}|x|_2^2 \right) \;=\; \frac{1}{2}|\xi|_2^2 - \frac{1}{2}|\xi - x|_2^2 \;\leq\; \frac{1}{2}|\xi|_2^2 .$$

On the other hand choosing $x := \xi$ we get

$$\sup_{x\in L^2(\mathbb{R})} \left(< x,\xi > -\tfrac{1}{2}|x|_2^2\right) \;\geq\; \tfrac{1}{2}|\xi|_2^2.$$

Hence $I(x) = \Lambda^*(x)$.

(iv) To show that E is dense in $\mathcal{S}'(\mathbb{R})$ we show that $\mathcal{S}(\mathbb{R}) \subseteq E$.
For given $\eta \in \mathcal{S}(\mathbb{R})$ choose $\xi := \eta$, then

$$< \eta,\xi > -\Lambda^*(\eta) \;=\; \tfrac{1}{2}|\eta|_2^2$$

and for any $z \in L^2(\mathbb{R})$, $z \neq \eta$ we have with $\xi = \eta$

$$< z,\xi > -\Lambda^*(z) \;=\; < z,\eta > -\tfrac{1}{2}|z|_2^2 \;=\; \tfrac{1}{2}|\eta|_2^2 - \tfrac{1}{2}|z-\eta|_2^2 \;\leq\; \tfrac{1}{2}|\eta|_2^2.$$

For $z \notin L^2(\mathbb{R})$ obviously $< z,\xi > -\Lambda^*(z) \;<\; \tfrac{1}{2}|\eta|_2^2$.
Furthermore $\Lambda(\gamma\xi) = \tfrac{1}{2}|\gamma\eta|_2^2 \;<\; \infty$ for all $\gamma > 1$.
Hence $\eta \in E$. □

The next result gives a generalization of the above theorem for one class of measures on $(\mathcal{S}'(\mathbb{R}),\mathcal{B})$:

Theorem 3.6 Let $\Phi : \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$ be of the form

$$\Phi \;=\; \sum_{i=1}^{n} c_i \exp\left(\sum_{j=1}^{m_i} < \cdot,\eta_{ij} >\right) \quad \text{with} \quad c_i \in \mathbb{R}^+, \eta_{ij} \in \mathcal{S}(\mathbb{R}).$$

Then obviously $\Phi \in (\mathcal{S})_+^*$ and for the corresponding measures $\{\nu_\varepsilon\}$ the strong large deviations principle holds with the good rate function

$$I(x) \;=\; \begin{cases} \tfrac{1}{2}|x|_2^2 & x \in L^2(\mathbb{R}) \\ \infty & \text{otherwise} \end{cases}.$$

Proof: In view of theorem 3.5 we only have to show that for Φ of the above form the measures $\{\nu_\varepsilon\} := \{(\nu_\Phi)_\varepsilon\}$ satisfy

$$\lim_{\varepsilon\to 0} \varepsilon \log \int_{\mathcal{S}'(\mathbb{R})} e^{< x,\frac{\xi}{\varepsilon} >}\, d\nu_\varepsilon(x) \;=\; \tfrac{1}{2}|\xi|_2^2.$$

First we show this result for Φ of the form $\Phi = \exp\left(\sum_{j=1}^{m} < \cdot,\eta_j >\right)$, $\eta_j \in \mathcal{S}(\mathbb{R})$. Then we have

$$\begin{aligned}
\int_{\mathcal{S}'(\mathbb{R})} e^{< x,\frac{\xi}{\varepsilon} >}\, d\nu_\varepsilon(x) \;&=\; \int_{\mathcal{S}'(\mathbb{R})} e^{< x,\frac{\xi}{\sqrt{\varepsilon}} >}\, d\nu(x) \\
&=\; \int_{\mathcal{S}'(\mathbb{R})} e^{< x,\frac{\xi}{\sqrt{\varepsilon}} >} \exp\left(\sum_{j=1}^{m} < x,\eta_j >\right) d\mu(x) \\
&=\; \exp\left(\frac{1}{2}\left|\frac{\xi}{\sqrt{\varepsilon}} + \sum_{j=1}^{m}\eta_j\right|_2^2\right)
\end{aligned}$$

and hence

$$\lim_{\varepsilon\to 0}\varepsilon\log\int\limits_{S'(\mathbb{R})} e^{<x,\frac{\xi}{\varepsilon}>}\,d\nu_\varepsilon(x) = \lim_{\varepsilon\to 0}\varepsilon\left[\frac{1}{2\varepsilon}|\xi|_2^2+\frac{1}{\sqrt{\varepsilon}}<\xi,\sum_{j=1}^m\eta_j>+\frac{1}{2}\left|\sum_{j=1}^m\eta_j\right|_2^2\right].$$
$$= \tfrac{1}{2}|\xi|_2^2$$

Obviously this result holds true if we replace Φ by $c\cdot\Phi$ with $c\in\mathbb{R}^+$.

So all that remains to show is that if the result is true for Φ_1 and Φ_2 it is also true for $\Phi_1+\Phi_2$. The rest then follows by induction.

Let $\{\nu_\varepsilon\}$ be the family of measures corrsponding to $\Phi_1+\Phi_2$, then

$$\varepsilon\log\int\limits_{S'(\mathbb{R})} e^{<x,\frac{\xi}{\varepsilon}>}\,d\nu_\varepsilon(x) = \varepsilon\log\left[<\Phi_1,e^{<\cdot,\frac{\xi}{\sqrt{\varepsilon}}>}>+<\Phi_2,e^{<\cdot,\frac{\xi}{\sqrt{\varepsilon}}>}>\right]$$

is bounded from below by

$$\varepsilon\log<\Phi_1,e^{<\cdot,\frac{\xi}{\sqrt{\varepsilon}}>}>$$

and from above by (assuming without loss of generality $<\Phi_1,e^{<\cdot,\frac{\xi}{\sqrt{\varepsilon}}>}>\geq<\Phi_2,e^{<\cdot,\frac{\xi}{\sqrt{\varepsilon}}>}>$)

$$\varepsilon\log 2<\Phi_1,e^{<\cdot,\frac{\xi}{\sqrt{\varepsilon}}>}>.$$

Since both boundaries converge for $\varepsilon\to 0$ to $\frac{1}{2}|\xi|_2^2$, so does

$$\lim_{\varepsilon\to 0}\varepsilon\log\int\limits_{S'(\mathbb{R})} e^{<x,\frac{\xi}{\varepsilon}>}\,d\nu_\varepsilon(x).$$

$\square$

Remark 3.7 Obviously the above result holds true if we replace the exponential by the Wick exponential

$$:\exp(<x,\eta>):\;:=\;\exp\left(<x,\eta>-\tfrac{1}{2}|\eta|_2^2\right)$$

and let η be an element of $L^2(\mathbb{R})$. But $:\exp(<\cdot,\eta>):$ is nothing else than the Radon-Nykodym derivative of the translated measure $\tau_\eta\mu:=\mu(\cdot+\eta)$ with respect to μ (cf. e.g. [HKP 93]), hence we get the result that the family $\{(\tau_\eta\mu)_\varepsilon\}$ also satisfies the large deviations principle with the same rate function.

Also for the scalings $\sigma_\lambda\mu$ of the measure μ the large deviations principle holds, but with a different rate function:

Lemma 3.8 Define for given $\lambda\in\mathbb{R}_{\setminus\{0\}}$ the scaling $\sigma_\lambda\mu$ of μ by $\sigma_\lambda\mu(A):=\mu(\lambda^{-1}A)$ for $A\in\mathcal{B}$.

Then for the family $\{\nu_\varepsilon\}=\{(\sigma_\lambda\mu)_\varepsilon\}$ the strong large deviations principle holds with the good rate function

$$I(x) = \begin{cases}\frac{1}{2\lambda^2}|x|_2^2 & x\in L^2(\mathbb{R})\\ \infty & \text{otherwise}\end{cases}.$$

Proof: Since $\sigma_\lambda \mu$ corresponds to an $(\mathcal{S})_+^*$-element (cf. e.g. [HKP 93], theorem 4.35), exponential tightness is given by theorem 3.3.

Analogously to the proof of theorem 3.5 we show

(i) For all $\xi \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}\Lambda(\xi) &= \lim_{\varepsilon \to 0} \varepsilon \log \int\limits_{\mathcal{S}'(\mathbb{R})} e^{<x, \frac{\xi}{\varepsilon}>} \, d\nu_\varepsilon(x) \\ &= \lim_{\varepsilon \to 0} \varepsilon \log \int\limits_{\mathcal{S}'(\mathbb{R})} e^{<x, \frac{\lambda\xi}{\sqrt{\varepsilon}}>} \, d\mu(x) \\ &= \tfrac{1}{2}\lambda^2 |\xi|_2^2\end{aligned}$$

(ii) If we define

$$I(x) = \begin{cases} \frac{1}{2\lambda^2}|x|_2^2 & x \in L^2(\mathbb{R}) \\ \infty & \text{otherwise} \end{cases}$$

then by choosing $x := \lambda^2 \xi$ we get

$$\sup_{x \in \mathcal{S}'(\mathbb{R})} \left(< x, \xi > - \frac{1}{2\lambda^2}|x|_2^2\right) \geq \tfrac{1}{2}\lambda^2 |\xi|_2^2.$$

On the other hand we have for all $x \in L^2(\mathbb{R})$

$$\begin{aligned}< x, \xi > - \frac{1}{2\lambda^2}|x|_2^2 &= < \frac{x}{\lambda}, \lambda\xi > - \frac{1}{2\lambda^2}|x|_2^2 \\ &= -\tfrac{1}{2}|\tfrac{x}{\lambda} - \lambda\xi|_2^2 + \tfrac{1}{2}\lambda^2|\xi|_2^2 \\ &\leq \tfrac{1}{2}\lambda^2|\xi|_2^2.\end{aligned}$$

Hence by lemma 3.4 $I(x) = \Lambda^*(x)$.

(iii) For given $\eta \in \mathcal{S}(\mathbb{R})$ choosing $\xi := \frac{1}{\lambda^2}\eta \in \mathcal{S}(\mathbb{R})$ we get

$$< \eta, \xi > - \Lambda^*(\eta) = \frac{1}{2\lambda^2}|\eta|_2^2$$

and for any $z \in L^2(\mathbb{R})$, $z \neq \eta$ we get for this ξ

$$\begin{aligned}< z, \xi > - \Lambda^*(z) &= \frac{1}{\lambda^2} < z, \eta > - \frac{1}{2\lambda^2}|z|_2^2 \\ &= -\frac{1}{2\lambda^2}|z - \eta|_2^2 + \frac{1}{2\lambda^2}|\eta|_2^2 \\ &< \frac{1}{2\lambda^2}|\eta|_2^2.\end{aligned}$$

Furthermore $\Lambda(\gamma\xi) = \frac{1}{2}\lambda^2\left|\frac{\gamma}{\lambda^2}\eta\right|_2^2 = \frac{\gamma^2}{2\lambda^2}|\eta|_2^2 < \infty$ for all $\gamma > 1$.
Hence η is an exposed point. □

However the rate function need not always look so nice but can be rather degenerated as the following example shows:

Example 3.9 For $y \in \mathcal{S}'(\mathbb{R})$ define $< \delta_y, \varphi >= \widetilde{\varphi}(y)$ for $\varphi \in (\mathcal{S})$. Then obviously δ_y belongs to $(\mathcal{S})^*_+$ and for the corresponding measures we can show directly that they satisfy the strong large deviations principle with the good rate function

$$I(x) \;=\; \begin{cases} 0 & x = 0 \\ \infty & \text{otherwise} \end{cases}.$$

Trivially I is a good rate function and for $H \subseteq \mathcal{S}'(\mathbb{R})$

$$\inf_{x \in H} I(x) \;=\; \begin{cases} 0 & 0 \in H \\ \infty & 0 \notin H \end{cases}$$

holds. Furthermore we have for $H \subseteq \mathcal{S}'(\mathbb{R})$

$$\nu_\varepsilon(H) \;=\; \begin{cases} 1 & \sqrt{\varepsilon} y \in H \\ 0 & \text{otherwise} \end{cases}.$$

So for $G \subseteq \mathcal{S}'(\mathbb{R})$ open and $0 \in G$ we have

$$\underline{\lim}_{\varepsilon \to 0}\, \varepsilon \log \nu_\varepsilon(G) \;=\; 0 \;=\; -\inf_{x \in G} I(x)$$

because also a small neighbourhood of 0 belongs to G and for $0 \notin G$

$$\underline{\lim}_{\varepsilon \to 0}\, \varepsilon \log \nu_\varepsilon(G) \;\geq\; -\inf_{x \in G} I(x)$$

holds trivially.
For $F \subseteq \mathcal{S}'(\mathbb{R})$ closed and $0 \notin F$ we have

$$\overline{\lim}_{\varepsilon \to 0}\, \varepsilon \log \nu_\varepsilon(F) \;=\; -\infty \;=\; -\inf_{x \in F} I(x)$$

because also a small neighbourhood of 0 doesn't belong to F and for $0 \in F$

$$\overline{\lim}_{\varepsilon \to 0}\, \varepsilon \log \nu_\varepsilon(F) \;\leq\; -\inf_{x \in F} I(x)$$

holds trivially because of $\nu_\varepsilon(F) \leq 1$.

Let us finally remark that the problem of calculating $\Lambda(\xi)$ is closely related to the calculation of certain limits of the $\mathcal{S}$-transform of Φ, defined as

$$\mathcal{S}\Phi(\xi) \;=\; < \Phi, :e^{<\cdot,\xi>}: > .$$

Namely it holds

$$\lim_{\varepsilon \to 0} \varepsilon \log \int\limits_{\mathcal{S}'(\mathbb{R})} e^{<x,\frac{\xi}{\varepsilon}>}\, d\nu_\varepsilon(x) \;=\; \tfrac{1}{2}|\xi|_2^2 + \lim_{\varepsilon \to 0} \varepsilon \log \mathcal{S}\Phi\Big(\frac{1}{\sqrt{\varepsilon}}\xi\Big)$$

each of the limits existing if the other one does.

As an easy application of the Potthoff-Streit characterization theorem (cf. [PS 91, KLP 94]) we therefore get that if the limit exists it has to be finite:

For $\Phi \in (\mathcal{S})^*$ there exist $c > 0$ and $p \in \mathbb{N}_0$ such that

$$|\mathcal{S}\Phi(\lambda\xi)| \;\leq\; c\, e^{\frac{1}{2}|\lambda|^2|\xi|_{-p}^2} \quad \text{for all } \lambda \in \mathbb{C} \text{ and } \xi \in \mathcal{S}(\mathbb{R}).$$

Hence if the limits exist we can estimate

$$\begin{aligned}\lim_{\varepsilon\to 0} \varepsilon \log \int_{\mathcal{S}'(\mathbb{R})} e^{<x,\frac{\xi}{\varepsilon}>}\, d\nu_\varepsilon(x) \;&=\; \tfrac{1}{2}|\xi|_2^2 + \lim_{\varepsilon\to 0} \varepsilon \log \mathcal{S}\Phi\Big(\frac{1}{\sqrt{\varepsilon}}\xi\Big) \\ &\leq\; \tfrac{1}{2}|\xi|_2^2 + \tfrac{1}{2}|\xi|_{-p}^2\end{aligned}.$$

Finally we can give another class of functionals on $\mathcal{S}'(\mathbb{R})$ for which the large deviations principle holds:

Theorem 3.10 Let $\Phi \in (\mathcal{S})_+^*$ such that the function $\lambda \to \mathcal{S}\Phi(\lambda\xi)$ is of order strictly less than 2 for any $\xi \in \mathcal{S}(\mathbb{R})$, that is there exist $c_1, c_2 > 0$, $p \in \mathbb{N}_0$ and $\delta < 2$ with

$$|\mathcal{S}\Phi(\lambda\xi)| \;\leq\; c_1\, e^{c_2|\lambda|^\delta|\xi|_{-p}^\delta} \quad \text{for all } \lambda \in \mathbb{C} \text{ and } \xi \in \mathcal{S}(\mathbb{R}).$$

Then for the corresponding measures $\{\nu_\varepsilon\}$ the strong large deviations principle holds with the good rate function

$$I(x) \;=\; \begin{cases} \frac{1}{2}|x|_2^2 & x \in L^2(\mathbb{R}) \\ \infty & \text{otherwise} \end{cases}.$$

Proof: Again we only have to show that

$$\lim_{\varepsilon\to 0} \varepsilon \log \int_{\mathcal{S}'(\mathbb{R})} e^{<x,\frac{\xi}{\varepsilon}>}\, d\nu_\varepsilon(x) \;=\; \tfrac{1}{2}|\xi|_2^2.$$

But with the assumed bound for $|\mathcal{S}\Phi(\lambda\xi)|$ this is an easy computation.

References.

[Cr 38] Cramér, H.: Sur un nouveau théorème-limite de la théorie des probabilités; *Actualités Scientifiques et Industrielles* **736** (1938) 2-23

[DS 89] Deuschel, J.D. and Stroock, D.W.: *Large Deviations.* San Diego: Academic Press (1989)

[DV 75a] Donsker, M.D. and Varadhan, S.R.S.: Asymtotic evaluation of certain Markov process expectations for large time I; *Comm. Pure Appl. Math.* **28** (1975) 1-47

[DV 75b] Donsker, M.D. and Varadhan, S.R.S.: Asymtotic evaluation of certain Markov process expectations for large time II; *Comm. Pure Appl. Math.* **28** (1975) 279-301

[DV 76] Donsker, M.D. and Varadhan, S.R.S.: Asymtotic evaluation of certain Markov process expectations for large time III; *Comm. Pure Appl. Math.* **29** (1976) 389-461

[DV 83] Donsker, M.D. and Varadhan, S.R.S.: Asymtotic evaluation of certain Markov process expectations for large time IV; *Comm. Pure Appl. Math.* **36** (1983) 183-212

[DZ 93] Dembo, A. and Zeitouni, O.: *Large Deviation Techniques and Applications.* Boston: Jones and Bartlett (1993)

[El 85] Ellis, R.S.: *Entropy, Large Deviations, and Statistical Mechanics.* New York: Springer-Verlag (1985)

[El 95] Ellis, R.S.: An Overview of the Theory of Large Deviations and Applications to Statistical Mechanics; *Scand. Actuarial J.* **1** (1995) 97-142

[HKP 93] Hida, T., Kuo, H.-H., Potthoff, J. and Streit, L.: *White noise – An Infinite Dimensional Calculus.* Dordrecht: Kluwer (1993)

[KLP 94] Kondratiev, Y., Leukert, P., Potthoff, J., Streit, L. and Westerkamp, W.: Generalized functionals in Gaussian spaces – the characterization theorem revisited; *Preprint* (1994)

[KS 76] Kondratiev, Y. and Samoilenko, Y.S.: Integral representation of generalized positive definite kernels of an infinite number of variables; *Soviet math. Dokl.* **17** (1976) 517-521

[Mi 62] Minlos, R.A.: Generalized random processes and their extension to a measure; *Sel. Transl. in Math. Statist. and Probability* Vol.3 (1962) 291-313

[PS 91] Potthoff, J. and Streit, L.: A characterization of Hida distributions; *J. Funct. Anal.* **101** (1991) 212-229

[Tr 67] Trèves, F.: *Topological Vector Spaces, Distributions and Kernels* New York, London: Academic Press (1967)

[Yo 90] Yokoi, Y.: Positive generalized white noise functionals; *Hiroshima math. J.* **20** (1990) 137-157