

**Invariant Star Products of Wick Type:  
Classification and Quantum Momentum Mappings**

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# Invariant Star Products of Wick Type: Classification and Quantum Momentum Mappings

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## **Abstract**

We extend our investigations on  $\mathfrak{g}$ -invariant Fedosov star products and quantum momentum mappings [22] to star products of Wick type on pseudo-Kähler manifolds. Star products of Wick type can be completely characterized by a local description as given by Karabegov in [16] for star products with separation of variables. We separately treat the action of a Lie group  $G$  on  $\mathcal{C}^\infty(M)[[\nu]]$  by (pull-backs with) diffeomorphisms and the action of a Lie algebra  $\mathfrak{g}$  on  $\mathcal{C}^\infty(M)[[\nu]]$  by (Lie derivatives with respect to) vector fields. Within Karabegov's framework, we prove necessary and sufficient conditions for a given star product of Wick type to be invariant in the respective sense. Moreover, our results yield a complete classification of invariant star products of Wick type. We also prove a necessary and sufficient condition for (the Lie derivative with respect to) a vector field to be even a quasi-inner derivation of a given star product of Wick type. We then transfer our former results about quantum momentum mappings for  $\mathfrak{g}$ -invariant Fedosov star products to the case of invariant star products of Wick type.

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## 1 Introduction

The concept of deformation quantization as introduced in the pioneering articles [2] by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer has proved to be an extremely useful framework for the problem of quantization. For recent general presentations of the topic see for example the articles by Gutt [12], Sternheimer [25], and Dito and Sternheimer [8] and the references therein for details.

There are various notions of invariance of star products with respect to Lie group resp. Lie algebra actions which were introduced and discussed by Arnal, Cortet, Molin, and Pinczon in [1]. In [3] Bertelson, Bieliavsky, and Gutt proved that the equivalence classes of  $G$ -invariant star products on a symplectic manifold are in bijection to formal series of  $G$ -invariant classes in the second deRham cohomology of the manifold. Another important notion in deformation quantization – which is one of the key ingredients of the formulation of phase space reduction in this framework (cf. [10, 5]) – is that of a quantum momentum mapping, an object introduced and studied in detail by Xu [27]. In [22], we studied  $\mathfrak{g}$ -invariant Fedosov star products and quantum momentum mappings and we gave necessary and sufficient conditions for the existence of quantum momentum mappings for these star products which particularly showed that generally, the existence of a classical momentum mapping does not imply the existence of a quantum momentum mapping. Some of the statements of [22] had already been presented in [13] and have since been published in [14].

The scope of the present letter is on the special kind of star products called star products of Wick type. Star products of Wick type are an important example of star products compatible with an additional geometric structure, namely the complex structure. They are closely related to geometric quantization [26] with a complex polarization, and to Berezin's quantization on Kähler manifolds. The star products constructed in that context by Moreno [21] and by Cahen, Gutt, and Rawnsley [7] are concrete examples. Moreover, star products of Wick type also appear as asymptotic expansions of the Berezin-Toeplitz quantization, cf. [4], and for details [24, 19].

We begin by recalling the definition of these special star products. On a pseudo-Kähler manifold  $(M, \omega, I)$  a star product is said to be of Wick type if the bidifferential operators determining the star product contain only derivatives in holomorphic directions in the first argument and only derivatives in anti-holomorphic directions in the second argument. This definition of star products of Wick type clearly is equivalent to the condition that on every open subset  $U \subseteq M$  star-right-multiplication with functions that are holomorphic on  $U$  and star-left-multiplication with functions that are anti-holomorphic on  $U$  both coincide with pointwise multiplication. Star products of Wick type for general Kähler manifolds were independently constructed by Karabegov in his work on star products with separation of variables [16, 17, 18] and by Bordemann and Waldmann [6] using a modified Fedosov construction with a fibrewise Wick product. The latter results have been generalized in [23], where it is shown under which conditions a generalized Fedosov star product constructed using a fibrewise Wick product is of Wick type and that in fact all star products of

Wick type are generalized Fedosov star products. Observe that star products of Wick type are star products with separation of variables in Karabegov's sense, but with interchanged rôles of holomorphic and anti-holomorphic derivatives. All the results in this letter shall be derived within Karabegov's framework of star products of Wick type.

The present letter is organized as follows: In Section 2, we briefly review some of Karabegov's main results as needed in the sequel. We also collect some notational conventions and shorthands.

In Section 3, we use Karabegov's results to independently treat the cases of invariance of a given star product of Wick type on a pseudo-Kähler manifold with respect to the action of Lie groups and Lie algebras. We give a complete classification of invariant star products of Wick type which shows that for star products of Wick type an even stronger result than the one of [3] for general invariant star products on a symplectic manifold holds. Eventually, we give a necessary and sufficient condition for a vector field  $X$  to be a quasi-inner derivation (see below for a definition).

With the results from Section 3, our statements about quantum momentum mappings from [22] are then adapted to the case of invariant star products of Wick type in Section 4.

**Conventions:** Whenever we speak of Kähler manifolds  $(M, \omega, I)$ , we more generally mean pseudo-Kähler manifolds since positivity of the Hermitean metric  $\omega(\cdot, I\cdot)$  is nowhere required. By  $\mathcal{C}^\infty(M)$ , we denote the complex-valued smooth functions and similarly  $\Gamma^\infty(TM)$  stands for the complex-valued smooth vector fields et cetera. Slightly abusing notation we denote elements of  $\Gamma^\infty(TM)$  and the corresponding derivations on  $\mathcal{C}^\infty(M)$  by the same symbol. We use Einstein's summation convention in local expressions.

## 2 Karabegov's Description and Characterization of Star Products of Wick Type

Let  $(M, \omega, I)$  be a Kähler manifold of real dimension  $2n$  with symplectic form  $\omega$  and complex structure  $I$ . In a local holomorphic chart, we write  $Z_k := \partial_{z^k}$  and  $\bar{Z}_l := \partial_{\bar{z}^l}$  for local basis vector fields of type  $(1, 0)$  and of type  $(0, 1)$  that locally span the  $+i$  and  $-i$  eigenspaces  $TM^{1,0}$  and  $TM^{0,1}$  of the complex structure  $I$ . For vector fields  $X \in \Gamma^\infty(TM) = \Gamma^\infty(TM^{1,0}) \oplus \Gamma^\infty(TM^{0,1})$ , we sometimes write  $X = \chi + \underline{\chi}$  with  $\chi \in \Gamma^\infty(TM^{1,0})$  and  $\underline{\chi} \in \Gamma^\infty(TM^{0,1})$ .

A star product of Wick type on  $(M, \omega, I)$  then is by definition a star product given for all  $f, g \in \mathcal{C}^\infty(M)$  by  $f \star g = \sum_{r=0}^{\infty} \nu^r C_r(f, g)$ , where, using some coefficient functions  $C_r^{K;\bar{L}}$ , in local holomorphic coordinates for  $r \geq 1$  each bidifferential operator has the form  $C_r(f, g) = \sum_{K, \bar{L}} C_r^{K;\bar{L}} \frac{\partial^{|\bar{K}|} f}{\partial z^{\bar{K}}} \frac{\partial^{|\bar{L}|} g}{\partial \bar{z}^{\bar{L}}}$ .

We shall now use Karabegov's results from [16] about star products with separation of variables to give a unique description of all star products of Wick type. Please note that the star products with separation of variables in Karabegov's original works [16, 17, 18] differ from the star products of Wick type considered here by a sign in the Poisson bracket and by an interchange of the rôles of holomorphic and anti-holomorphic coordinates. In the sequel, we shall therefore adapt Karabegov's results to our setting.

Let now  $\star$  be a star product of Wick type on  $(M, \omega, I)$ . Then (cf. [16, Lemma 2]) there are formal functions  $u_k, \bar{v}_l \in \mathcal{C}^\infty(U)[[\nu]]$  defined on an open, contractible domain  $U \subseteq M$  of a local holomorphic chart  $(z, U)$  of  $M$  such that

$$f \star u_k = f u_k + \nu Z_k(f) \quad \text{and} \quad \bar{v}_l \star f = \bar{v}_l f + \nu \bar{Z}_l(f) \quad (1)$$

for all  $f \in \mathcal{C}^\infty(U)[[\nu]]$ . We shall always reserve the symbols  $u_k$  and  $\bar{v}_l$  to denote functions as in (1). Moreover, Karabegov considers locally defined formal series of one-forms  $\alpha, \beta \in \Gamma^\infty(T^*U)[[\nu]]$  given by  $\alpha := -u_k dz^k$  which is of type  $(1, 0)$  and  $\beta := \bar{v}_l d\bar{z}^l$  which is of type  $(0, 1)$ . As the  $\star$ -right-multiplication with  $u_k$  obviously commutes with the  $\star$ -left-multiplication with  $\bar{v}_l$ , one in addition

obtains from (1) that  $\bar{\partial}\alpha = \partial\beta$ . One can show that this procedure yields a formal series of closed two-forms of type  $(1, 1)$  on  $M$ . In the following the so-defined formal two-form that can be associated to any star product  $\star$  of Wick type shall be denoted by  $K(\star)$  and is referred to as Karabegov's characterizing form of the star product  $\star$ . It is easy to see from the very definition that  $K(\star) \in \omega + \nu Z_{\text{dR}}^2(M, \mathbb{C})^{1,1}[[\nu]]$ , where  $\nu Z_{\text{dR}}^2(M, \mathbb{C})^{1,1}[[\nu]] = \{\Omega \in \nu\Gamma^\infty(\wedge^2 T^*M)[[\nu]] \mid d\Omega = 0, \pi^{1,1}\Omega = \Omega\}$ , and hence by the  $\partial$ - $\bar{\partial}$ -Poincaré Lemmas, there exist local formal functions  $\varphi \in \mathcal{C}^\infty(U)[[\nu]]$  on every contractible domain  $U$  of holomorphic coordinates such that  $K(\star)|_U = \partial\bar{\partial}\varphi$  and  $\varphi$  is called a formal local Kähler potential of  $K(\star)$ . With such a formal local Kähler potential the equations

$$f \star Z_k(\varphi) = f Z_k(\varphi) + \nu Z_k(f) \quad \text{and} \quad \bar{Z}_l(\varphi) \star f = \bar{Z}_l(\varphi)f + \nu \bar{Z}_l(f) \quad (2)$$

hold for all  $f \in \mathcal{C}^\infty(U)[[\nu]]$ . Together, one finds:

**Theorem 2.1** ([16, Thm. 1]) *Let  $\star$  be a star product of Wick type on a Kähler manifold  $(M, \omega, I)$ . Then  $K(\star) \in \omega + \nu Z_{\text{dR}}^2(M, \mathbb{C})^{1,1}[[\nu]]$  associates a formal series of closed two-forms of type  $(1, 1)$  on  $M$  – which is a deformation of the Kähler form  $\omega$  – to this star product. In case  $\varphi \in \mathcal{C}^\infty(U)[[\nu]]$  is a formal local Kähler potential of  $K(\star)$ , the equations (2) hold for all  $f \in \mathcal{C}^\infty(U)[[\nu]]$ .*

Conversely, in [16, Sect. 4] Karabegov has shown that to each such form  $K$  as in the preceding theorem one can assign a star product  $\star$  of Wick type such that the characterizing form  $K(\star)$  of this star product actually coincides with this given  $K$ . To this end Karabegov has given an explicit construction of such a star product extensively using local considerations, and he proved:

**Theorem 2.2** ([16, Thm. 2]) *Karabegov's characterizing form induces a bijection*

$$\{ \text{star products of Wick type on } (M, \omega, I) \} \ni \star \mapsto K(\star) - \omega \in \nu Z_{\text{dR}}^2(M, \mathbb{C})^{1,1}[[\nu]] \quad (3)$$

*between star products of Wick type on  $(M, \omega, I)$  and formal closed two-forms on  $M$  of type  $(1, 1)$  with formal degree  $\geq 1$ .*

In [23, Thm. 5.2] an alternative proof of the fact that a star product of Wick type is completely determined by one of the equations (2) valid in every holomorphic chart and hence that it is determined by its characterizing form has been given.

### 3 Classification of Invariant Star Products of Wick Type

We are going to treat the cases of invariance under group actions on  $\mathcal{C}^\infty(M)[[\nu]]$  by diffeomorphisms and invariance with respect to Lie algebra actions on  $\mathcal{C}^\infty(M)[[\nu]]$  by vector fields separately and independently. That way, we do not have to assume that the groups considered are connected, and we do not need to restrict to infinitesimal actions of groups.

Note that, in the case of a Lie group  $G$ , the statements for the sufficient conditions for an ordinary Fedosov star product as defined in [9] to be  $G$ -invariant are well-known (cf. [3, Sect. 3]). In [22], we have proved a statement giving both necessary and sufficient conditions for generalized Fedosov star products to be  $\mathfrak{g}$ -invariant. Note that owing to the fact that every star product of Wick type can be obtained using the generalized Fedosov construction on Kähler manifolds [23], one could as well use a Fedosov construction to prove the statements of the following sections. In particular, the proofs of the sufficient conditions for invariance with respect to a diffeomorphism and invariance with respect to a vector field in Propositions 3.1 and 3.2 are straightforward using Fedosov methods. We only mention this fact and we shall not go into details but rather treat everything within Karabegov's description of star products of Wick type.

First we shall give necessary and sufficient conditions for a diffeomorphism of  $M$  to induce – via pull-back – an automorphism of a given star product of Wick type.

**Proposition 3.1** *Let  $\star$  be a star product of Wick type on  $(M, \omega, I)$  with Karabegov's characterizing form  $K(\star)$  and let  $\phi$  be a diffeomorphism of  $M$ . Then the pull-back  $\phi^*$  is an automorphism of  $\star$  if and only if*

$$\phi^*I = I \quad \text{and} \quad \phi^*K(\star) = K(\star). \quad (4)$$

PROOF: Let us first prove that  $\phi^*I = I$  and  $\phi^*K(\star) = K(\star)$  are necessary conditions for  $\phi^*$  to be an automorphism of  $\star$ . Then  $\phi^{-1*}$  also being an automorphism of  $\star$  and Karabegov's relations (1) imply that on  $\phi(U)$  we have

$$f \star \phi^{-1*}u_k = \phi^{-1*}(\phi^*f \star u_k) = \phi^{-1*}(\phi^*f u_k + \nu Z_k(\phi^*f)) = f \phi^{-1*}u_k + \nu(\phi^{-1*}Z_k)(f) \quad (5)$$

for all  $f \in C^\infty(\phi(U))$ . In particular, we can choose a local holomorphic chart  $(z', \phi(U))$ , and then for  $f = \bar{z}'^l$  the last equation becomes  $(\phi^{-1*}Z_k)(\bar{z}'^l) = 0$  which implies that  $\phi^{-1*}Z_k$  is still of type  $(1, 0)$ , that is  $I(\phi^{-1*}Z_k) = i\phi^{-1*}Z_k$ . But this implies that  $\phi^{-1*}$  maps vector fields of type  $(1, 0)$  to vector fields of type  $(1, 0)$  and hence  $\phi^*I = I$ . Particularly,  $\phi^*$  maps local holomorphic charts to local holomorphic charts. Now we calculate  $K(\star)$  and  $\phi^*K(\star)$ . Using equation (5) for  $\phi^{-1}$  replaced by  $\phi$  we obtain  $f \star \phi^*u_k = f\phi^*u_k + \nu(\phi^*Z_k)(f)$  for  $f \in C^\infty(\phi^{-1}(U))$ . Since  $\phi^*Z_k = Z'_k$  in the holomorphic chart  $(z', U') = (\phi^*z, \phi^{-1}(U))$ , we can use  $\phi^*u_k$  to calculate  $K(\star)$  in the chart  $(z', U')$ :

$$\begin{aligned} K(\star)|_{U'} &= -\bar{\partial}(\phi^*u_k dz'^k) = -\bar{Z}'_l(\phi^*u_k) d\bar{z}'^l \wedge dz'^k = -\phi^*(\bar{Z}_l u_k)(\phi^*d\bar{z}^l) \wedge (\phi^*dz^k) \\ &= \phi^*(-\bar{\partial}(u_k dz^k)) = \phi^*(K(\star)|_U). \end{aligned}$$

Please note that analogously, we could have used the equation  $\phi^*\bar{v}_l \star g = \phi^*\bar{v}_l g + \nu(\phi^*\bar{Z}_l)(g)$  for  $g \in C^\infty(\phi^{-1}(U))$  to obtain the same result.

Conversely, let us now assume that  $\phi^*I = I$  and that  $\phi^*K(\star) = K(\star)$ . We want to prove that this implies that  $\phi^*(f \star g) = \phi^*f \star \phi^*g$  for all formal functions  $f, g \in C^\infty(M)[[\nu]]$ . For this purpose, consider the star product

$$f \star' g := \phi^{-1*}(\phi^*f \star \phi^*g).$$

Since  $\phi^{-1*}$  is type-preserving and since  $\phi^{-1*}K(\star) = K(\star)$ , which particularly implies  $\phi^{-1*}\omega = \omega$ , we obviously find that  $\star'$  is a star product of Wick type on  $(M, \omega, I)$ . But star products of Wick type are uniquely determined by their Karabegov form  $K$ . Therefore the proof is done if we can show that  $K(\star') = K(\star)$  since this implies that  $\star'$  equals  $\star$  and hence that  $\phi^*$  is an automorphism of  $\star$ . To see that this is true, we write  $\phi^{-1*}Z_k = \tilde{Z}_k$  in the holomorphic chart  $(\tilde{z}, \tilde{U}) = (\phi^{-1*}z, \phi(U))$  and compute as in equation (5):  $f \star' \phi^{-1*}u_k = f\phi^{-1*}u_k + \nu\tilde{Z}_k(f)$  for  $f \in C^\infty(\phi(U))$ . Therefore, we have

$$K(\star')|_{\phi(U)} = -\bar{\tilde{Z}}_l(\phi^{-1*}u_k) d\tilde{z}^l \wedge d\tilde{z}^k = \phi^{-1*}(-\bar{Z}_l u_k d\bar{z}^l \wedge dz^k) = \phi^{-1*}(K(\star)|_U) = K(\star)|_{\phi(U)},$$

where we used  $K(\star) = \phi^*K(\star)$  to obtain the last equality.  $\square$

Now we shall give necessary and sufficient conditions for a vector field on  $M$  to define – via the Lie derivative – a derivation of a given star product of Wick type.

**Proposition 3.2** *Let  $\star$  be a star product of Wick type on  $(M, \omega, I)$  with Karabegov's characterizing form  $K(\star)$  and let  $X \in \Gamma^\infty(TM)$  be a vector field on  $M$ . Then  $X$  is a derivation of  $\star$  if and only if*

$$\mathcal{L}_X I = 0 \quad \text{and} \quad \mathcal{L}_X K(\star) = 0. \quad (6)$$

PROOF: Let us first show that  $\mathcal{L}_X I = 0 = \mathcal{L}_X K(\star)$  implies that  $X$  is a derivation of  $\star$ . We write  $X = \chi + \underline{\chi}$  with  $\chi \in \Gamma^\infty(TM^{1,0})$  and  $\underline{\chi} \in \Gamma^\infty(TM^{0,1})$ , locally  $X = \chi^n Z_n + \chi^{\bar{l}} \bar{Z}_l$ . Then we get from  $0 = (\mathcal{L}_X I)Z_k$  that  $\mathcal{L}_X Z_k$  must be of type  $(1, 0)$ . Since locally  $\mathcal{L}_X Z_k = -\left(Z_k(\chi^n)Z_n + Z_k(\chi^{\bar{l}})\bar{Z}_l\right)$ , this implies that  $Z_k(\chi^{\bar{l}}) = 0$ , which means that  $\chi^{\bar{l}}$  is locally anti-holomorphic. Likewise, from  $(\mathcal{L}_X I)\bar{Z}_m = 0$  we find that  $\chi^n$  is locally holomorphic. Now since  $K(\star)$  is closed we have  $0 = \mathcal{L}_X K(\star) = di_X K(\star)$ , i.e.  $i_X K(\star)$  is a closed one-form on

$M$ . Hence on an open, contractible domain  $U$  of a chart there exists a local formal function  $f \in \mathcal{C}^\infty(U)[[\nu]]$  with  $i_X K(\star)|_U = df$ , hence  $\partial f = i_X K(\star)|_U$  and  $\bar{\partial} f = i_X K(\star)|_U$ . Using the first of these equations with  $K(\star)$  written out in local coordinates  $K(\star)|_U = Z_k(\bar{v}_l) dz^k \wedge d\bar{z}^l$  then gives  $Z_k(f) dz^k = -Z_k(\chi^{\bar{l}} \bar{v}_l) dz^k$ , which follows using that  $\chi^{\bar{l}}$  is locally anti-holomorphic. Therefore locally  $f = -\chi^{\bar{l}} \bar{v}_l + \bar{h}$  with a locally anti-holomorphic formal function  $\bar{h} \in \mathcal{C}^\infty(U)[[\nu]]$ . Then since  $\star$  is of Wick type and since  $\chi^{\bar{l}}$  is locally anti-holomorphic, we have  $\chi^{\bar{l}} \bar{v}_l = \chi^{\bar{l}} \star \bar{v}_l$ . Hence for all local functions  $g \in \mathcal{C}^\infty(U)$  the second of the equations (1) implies that the star product  $f \star g$  is given by

$$f \star g = (\bar{h} - \chi^{\bar{l}} \star \bar{v}_l) \star g = \bar{h}g - \chi^{\bar{l}}(\bar{v}_l g + \nu \bar{Z}_l(g)).$$

On the other hand, writing  $K(\star)|_U = -\bar{Z}_l(u_k) d\bar{z}^l \wedge dz^k$  and using  $\bar{\partial} f = i_X K(\star)|_U$ , an analogous calculation implies that  $f = \chi^k u_k + h$  with a locally holomorphic formal function  $h \in \mathcal{C}^\infty(U)[[\nu]]$ . Then as above for all  $g \in \mathcal{C}^\infty(U)$  we have

$$g \star f = gh + (gu_k + \nu Z_k(g)) \chi^k.$$

Combining these equations, we find

$$\text{ad}_\star(f)g = \bar{h}g - \chi^{\bar{l}}(\bar{v}_l g + \nu \bar{Z}_l(g)) - gh - (gu_k + \nu Z_k(g)) \chi^k.$$

But evaluating this equation for  $g = 1$ , we get  $0 = \bar{h} - \chi^{\bar{l}} \bar{v}_l - \chi^k u_k - h$ , and therefore  $\text{ad}_\star(f)g = -\nu(\chi^{\bar{l}} \bar{Z}_l + \chi^k Z_k)(g) = -\nu X(g)$ . But this means there is a local formal function  $f \in \mathcal{C}^\infty(U)[[\nu]]$  such that on  $U$  the Lie derivative is quasi-inner, that is

$$X|_{\mathcal{C}^\infty(U)} = -\frac{1}{\nu} \text{ad}_\star(f)|_{\mathcal{C}^\infty(U)}. \quad (7)$$

We shall later use this fact to prove a necessary and sufficient condition for  $X$  to be globally quasi-inner. But now we have finished the first part of the proof since (7) implies that  $X$  is a derivation of  $\star$ .

Conversely, let  $X$  be a derivation of  $\star$ . Then, applying  $X$  to the right-hand side of the first equation in (1), we have  $X(f \star u_k) = X(f u_k) + \nu X(Z_k(f))$ , but using the derivation property and the first equation of (1) we also have  $X(f \star u_k) = X(f) u_k + \nu Z_k(X(f)) + f \star X(u_k)$  and hence  $f \star X(u_k) = f X(u_k) + \nu[X, Z_k](f)$ . Now in a local holomorphic chart  $(z, U)$  of  $M$ , we choose  $f = \bar{z}^l$  and we write  $X$  in local holomorphic coordinates as  $X = \chi^m Z_m + \chi^{\bar{l}} \bar{Z}_l$ . First,  $\star$ -left-multiplication with  $f = \bar{z}^l$  is just the pointwise product, hence we get  $0 = \nu[X, Z_k](\bar{z}^l) = -Z_k(\chi^{\bar{l}})$ . Analogously, from the second equation of (1) one finds  $-\bar{Z}_l(\chi^m) = 0$ . Now we have in local holomorphic coordinates:  $(\mathcal{L}_X I)Z_k = i\mathcal{L}_X Z_k - I([\chi^m Z_m + \chi^{\bar{l}} \bar{Z}_l, Z_k])$ . But  $Z_k(\chi^{\bar{l}}) = 0$  implies  $[\chi^m Z_m + \chi^{\bar{l}} \bar{Z}_l, Z_k] = -Z_k(\chi^m) Z_m$ . Therefore  $I\mathcal{L}_X Z_k = i\mathcal{L}_X Z_k$  and hence  $(\mathcal{L}_X I)Z_k = 0$ . Likewise, one finds  $(\mathcal{L}_X I)\bar{Z}_l = 0$ . Together this implies  $\mathcal{L}_X I = 0$ . To show that  $\mathcal{L}_X K(\star) = 0$ , use that  $K(\star) = \bar{\partial}(-u_k dz^k)$  and that we have  $\mathcal{L}_X I = 0$ . Then  $\mathcal{L}_X K(\star) = -\bar{\partial}\mathcal{L}_X(u_k dz^k) = -\bar{\partial}(X(u_k) dz^k + u_k d\chi^k) = -(\bar{Z}_l(X(u_k)) d\bar{z}^l \wedge dz^k + \bar{Z}_l(u_m) Z_k(\chi^m) d\bar{z}^l \wedge dz^k)$ , where we have used that  $\bar{\partial}d\chi^k = 0$  since  $\chi^k$  is locally holomorphic. But this last expression equals zero, since applying  $X$  to  $\bar{v}_l \star u_k$ , again the derivation property and (1) yield  $[X, Z_k](\bar{v}_l) = \bar{Z}_l(X(u_k))$ . Explicitly this becomes  $-Z_k(\chi^m) \bar{Z}_l(u_m) = \bar{Z}_l(X(u_k))$  since  $[X, Z_k] = -Z_k(\chi^m) Z_m$ , and  $Z_k(\bar{v}_l) = \bar{Z}_l(u_k)$ . Hence  $\mathcal{L}_X K(\star) = 0$ , and the proposition is proved.  $\square$

We have now proved all the prerequisites we need to give the classification of the invariant star products of Wick type on a Kähler manifold. We first have to recall some definitions of notions of invariance of star products from [1].

Let  $G$  be a Lie group and let  $\Phi : G \times M \rightarrow M$  denote a (left-)action of  $G$  on  $M$ . Writing  $\phi_g \forall g \in G$  for the diffeomorphism of  $M$  defined by  $\phi_g(m) := \Phi(g, m) \forall m \in M$  one obviously has that  $r(g)f := \phi_{g^{-1}}^* f$  defines a Lie group action of  $G$  on  $\mathcal{C}^\infty(M)$  that naturally extends to a Lie group action on  $\mathcal{C}^\infty(M)[[\nu]]$ . Now recall that a star product  $\star$  on  $(M, \omega)$  is called  $G$ -invariant if  $r(g)$  is an automorphism of  $\star$  for all  $g \in G$ .

Furthermore, let  $\mathfrak{g}$  be a finite dimensional real or complex Lie algebra and let  $X : \mathfrak{g} \rightarrow \Gamma^\infty(TM) : \xi \mapsto X_\xi$  denote a Lie algebra anti-homomorphism. Then obviously  $\varrho(\xi)f := -X_\xi(f)$  defines a Lie algebra action of  $\mathfrak{g}$  on  $C^\infty(M)$  that also extends naturally to  $C^\infty(M)[[\nu]]$ . Also recall that a star product  $\star$  on  $(M, \omega)$  is called  $\mathfrak{g}$ -invariant if  $\varrho(\xi)$  is a derivation of  $\star$  for all  $\xi \in \mathfrak{g}$ .

Observe that from the action of a Lie group  $G$  one obtains a corresponding Lie algebra action of  $\mathfrak{g} = \text{Lie}(G)$  by  $X_\xi(m) := \left. \frac{d}{dt} \right|_{t=0} \Phi(\exp(t\xi), m)$  for all  $m \in M$  and in this case  $G$ -invariance of a star product clearly implies  $\mathfrak{g}$ -invariance. In case  $G$  is in addition connected one even has that  $\mathfrak{g}$ -invariance with respect to the above action implies  $G$ -invariance.

With the notations from above and from Propositions 3.1 and 3.2 we obtain the following classification result:

**Theorem 3.3** *Let  $(M, \omega, I)$  be a Kähler manifold.*

i.) *For a given Lie group action  $r$  as above there are  $G$ -invariant star products of Wick type on  $(M, \omega, I)$  if and only if  $\phi_g^* I = I$  and  $\phi_g^* \omega = \omega$  for all  $g \in G$ . In this case the set of  $G$ -invariant star products of Wick type is in bijection to  $\{\Omega \in \nu Z_{\text{dR}}^2(M, \mathbb{C})^{1,1}[[\nu]] \mid \phi_g^* \Omega = \Omega \forall g \in G\}$  and the bijection is given by the restriction of the mapping according to (3) to the respective  $G$ -invariant elements.*

ii.) *For a given Lie algebra action  $\varrho$  as above there are  $\mathfrak{g}$ -invariant star products of Wick type on  $(M, \omega, I)$  if and only if  $\mathcal{L}_{X_\xi} I = 0$  and  $\mathcal{L}_{X_\xi} \omega = 0$  for all  $\xi \in \mathfrak{g}$ . In this case the set of  $\mathfrak{g}$ -invariant star products of Wick type is in bijection to  $\{\Omega \in \nu Z_{\text{dR}}^2(M, \mathbb{C})^{1,1}[[\nu]] \mid \mathcal{L}_{X_\xi} \Omega = 0 \forall \xi \in \mathfrak{g}\}$  and the bijection is given by the restriction of the mapping according to (3) to the respective  $\mathfrak{g}$ -invariant elements.*

**Remark 3.4** *It is interesting to note that the necessary and sufficient conditions for the existence of  $G$ -invariant star products of Wick type on Kähler manifolds given above are precisely – besides other additional conditions that guarantee that the reduced phase space exists as a smooth symplectic manifold or more generally as a stratified symplectic space – the conditions that are sufficient for the reduced phase space to be a Kähler manifold or more generally a stratified Kähler space (cf. [15]).*

We now state a proposition giving a necessary and sufficient condition for a derivation of a star product of Wick type given by a vector field  $X$  to satisfy

$$X(g) = -\frac{1}{\nu} \text{ad}_\star(f)g \quad (8)$$

with some  $f \in C^\infty(M)[[\nu]]$  for all  $g \in C^\infty(M)[[\nu]]$ . In this case  $X$  is called a quasi-inner (or essentially inner [11]) derivation.

We would like to recall that again, since any given star product of Wick type can be realized as a certain Fedosov star product, one can alternatively employ the generalized Fedosov construction for star products of Wick type on Kähler manifolds [23] and prove, with several technical adaptations, the analogue of [22, Prop. 3.9]. Also note that the condition given in [22] was first presented for ordinary slightly more special Fedosov star products by Gutt in [13] (cf. also [14, Thm. 6.2]). The proof that the given condition is sufficient for these Fedosov star products was originally published in [20, Prop. 4.3].

However, we shall not use Fedosov methods in this letter and, as for the other statements in this section, we give a proof that only uses Karabegov's description of star products of Wick type.



**Proposition 3.5** (cf. [22, Prop. 3.9]) *Let  $\star$  be a star product of Wick type on  $(M, \omega, I)$  with Karabegov's characterizing form  $K(\star)$ . Assume that  $X \in \Gamma^\infty(TM)$  is a vector field such that  $X$  is a derivation of  $\star$ . Then  $X$  is even quasi-inner if and only if there is a formal function  $f \in \mathcal{C}^\infty(M)[[\nu]]$  such that*

$$df = i_X K(\star), \quad (9)$$

and then  $X(g) = X_{f_0}(g) = -\frac{1}{\nu} \text{ad}_\star(f)g$  for all  $g \in \mathcal{C}^\infty(M)[[\nu]]$ , where we have written  $f = f_0 + f_+$  with  $f_0 \in \mathcal{C}^\infty(M)$  and  $f_+ \in \nu \mathcal{C}^\infty(M)[[\nu]]$ .

PROOF: Let  $X$  be a derivation of  $\star$ , then from Proposition 3.2 we know that  $\mathcal{L}_X I = 0 = \mathcal{L}_X K(\star)$ . Now we have already shown in equation (7) that under these conditions on every open, contractible set  $U \subseteq M$  there is a local formal function  $f \in \mathcal{C}^\infty(U)[[\nu]]$  with  $df = i_X K(\star)|_U$  and that this implies that  $X|_{\mathcal{C}^\infty(U)} = -\frac{1}{\nu} \text{ad}_\star(f)|_{\mathcal{C}^\infty(U)}$ . Hence if there is a globally defined formal function  $f \in \mathcal{C}^\infty(M)[[\nu]]$  with  $df = i_X K(\star)$ , the proof of Proposition 3.2 also shows that  $X$  is quasi-inner.

Conversely let  $X = \chi + \underline{\chi}$ , where  $\chi \in \Gamma^\infty(TM^{1,0})$  and  $\underline{\chi} \in \Gamma^\infty(TM^{0,1})$ , be a quasi-inner derivation of  $\star$ , that is for all  $g \in \mathcal{C}^\infty(M)$  we have  $X(g) = -\frac{1}{\nu} \text{ad}_\star(f)g$ , or equivalently  $g \star f - \nu \chi(g) = f \star g + \nu \underline{\chi}(g)$ . Since  $\star$  is of Wick type, the left-hand side of the former equation contains only derivatives of  $g$  in holomorphic directions, while the right-hand side only contains derivatives in anti-holomorphic ones. This implies that  $g \star f - \nu \chi(g) = hg (= f \star g + \nu \underline{\chi}(g))$ , with a formal function  $h \in \mathcal{C}^\infty(M)[[\nu]]$  and for  $g = 1$  this particularly yields  $h = f$  and therefore we have  $\chi(g) = \frac{1}{\nu}(g \star f - gf)$  and  $\underline{\chi}(g) = \frac{1}{\nu}(fg - f \star g)$ . As in the previous proofs, we now use the equations (1) and calculate  $\underline{\chi}(u_k) = \frac{1}{\nu}(fu_k - (fu_k + \nu Z_k(f))) = -Z_k(f)$  and likewise  $\chi(\bar{v}_i) = \bar{Z}_i(f)$ . From these equations and the very definition of  $K(\star)$  it is obvious that  $\partial f = i_{\underline{\chi}} K(\star)$  and likewise  $\bar{\partial} f = i_{\chi} K(\star)$  and hence (9) holds and the necessary condition is proved. For the remaining statement of the proposition, just observe that the zeroth order in  $\nu$  implies that  $df_0 = i_X \omega$ , and hence  $X = X_{f_0}$  is a Hamiltonian vector field for the function  $f_0 \in \mathcal{C}^\infty(M)$ .  $\square$

## 4 Quantum Momentum Mappings for Invariant Star Products of Wick Type

We have now proved all the prerequisites we needed to prove our results on quantum momentum mappings in [22, Sect. 4]. Hence we can simply transfer our results thence to the case of star products of Wick type.

We shall first need to recall some definitions and notations from [27] and [22]. Then we state the analoga of the results proved in [22, Sect. 4]. For the sake of brevity, and since all the proofs are (almost) literally the same, we simply write down the statements themselves and refer the reader to [22, Sect. 4] for proofs and some additional comments.

We shall no longer treat the cases of Lie group actions and Lie algebra actions separately and simply speak of invariant star products in this section. Moreover in the case of a Lie group action we always consider the induced Lie algebra action by means of its infinitesimal generators.

Considering some complex vector space  $V$  endowed with a representation  $\pi : \mathfrak{g} \rightarrow \text{Hom}(V, V)$  of the Lie algebra  $\mathfrak{g}$  in  $V$  we denote the space of  $V$ -valued  $k$ -multilinear alternating forms on  $\mathfrak{g}$  by  $C^k(\mathfrak{g}, V)$  and the corresponding Chevalley-Eilenberg differential shall be denoted by  $\delta_\pi : C^\bullet(\mathfrak{g}, V) \rightarrow C^{\bullet+1}(\mathfrak{g}, V)$ . Moreover the spaces of the corresponding cocycles and coboundaries resp. the corresponding cohomology spaces shall be denoted by  $Z_\pi^k(\mathfrak{g}, V)$  and  $B_\pi^k(\mathfrak{g}, V)$  resp.  $H_\pi^k(\mathfrak{g}, V)$ .

A Lie group action  $r$  resp. a Lie algebra action  $\varrho$  is called Hamiltonian if and only if there is an element  $J_0 \in C^1(\mathfrak{g}, \mathcal{C}^\infty(M))$  such that  $X_{J_0(\xi)} = X_\xi$  for all  $\xi \in \mathfrak{g}$ , i.e.  $i_{X_\xi} \omega = dJ_0(\xi)$ . In this case  $\varrho(\xi) \cdot = \{J_0(\xi), \cdot\}$  and  $J_0$  is said to be a Hamiltonian for the action  $r$  resp.  $\varrho$ . In case  $J_0$  is equivariant with respect to the coadjoint representation of  $\mathfrak{g}$ , i.e.  $\{J_0(\xi), J_0(\eta)\} = J_0([\xi, \eta])$  for all

$\xi, \eta \in \mathfrak{g}$ , the Hamiltonian  $J_0$  is called a classical momentum mapping. For a Lie group action this is the case if in particular  $r(g)J_0(\xi) = J_0(\text{Ad}(g)\xi)$  for all  $g \in G$  and all  $\xi \in \mathfrak{g}$ .

Let now  $\star$  be an invariant star product, then  $J = J_0 + J_+ \in C^1(\mathfrak{g}, C^\infty(M))[[\nu]]$  with  $J_0 \in C^1(\mathfrak{g}, C^\infty(M))$  and  $J_+ \in \nu C^1(\mathfrak{g}, C^\infty(M))[[\nu]]$  is called a quantum Hamiltonian for the action  $r$  resp.  $\varrho$  in case

$$\varrho(\xi) = \frac{1}{\nu} \text{ad}_\star(J(\xi)) \quad \text{for all } \xi \in \mathfrak{g}. \quad (10)$$

$J$  is called a quantum momentum mapping if in addition

$$\frac{1}{\nu} (J(\xi) \star J(\eta) - J(\eta) \star J(\xi)) = J([\xi, \eta]) \quad (11)$$

for all  $\xi, \eta \in \mathfrak{g}$ .

The zeroth orders of (10) and (11) just mean that  $J_0$  is a classical momentum mapping.

Also recall the definition of a strongly invariant star product from [1]: Let  $J_0$  be a classical momentum mapping for the action  $r$  resp.  $\varrho$ . Then an invariant star product is called strongly invariant if and only if  $J = J_0$  defines a quantum Hamiltonian for this action.

We now collect the analoga for Wick type star products of our results presented in [22, Sect. 4].

As in [22, Ded. 4.4], the condition for the existence of a quantum Hamiltonian for an action  $r$  resp.  $\varrho$  can be seen directly from Proposition 3.5:

**Deduction 4.1** *An invariant star product  $\star$  of Wick type on  $(M, \omega, I)$  with Karabegov's characterizing form  $K(\star)$  admits a quantum Hamiltonian if and only if there is an element  $J \in C^1(\mathfrak{g}, C^\infty(M))[[\nu]]$  such that*

$$dJ(\xi) = i_{X_\xi} K(\star) \quad \forall \xi \in \mathfrak{g}, \quad (12)$$

*i.e. if and only if the cohomology class of  $i_{X_\xi} K(\star)$  vanishes for all  $\xi \in \mathfrak{g}$ . Moreover, from equation (12)  $J$  is determined (in case it exists) up to elements in  $C^1(\mathfrak{g}, \mathbb{C})[[\nu]]$ .*

We also have the following statement on strong invariance:

**Corollary 4.2** (cf. [22, Cor. 4.7]) *Let  $J_0$  be a classical momentum mapping for the action  $r$  resp.  $\varrho$ . Then an invariant star product of Wick type  $\star$  with Karabegov's characterizing form  $K(\star)$  is strongly invariant if and only if*

$$i_{X_\xi}(K(\star) - \omega) = 0 \quad \text{for all } \xi \in \mathfrak{g}. \quad (13)$$

*In this case every classical momentum mapping defines a quantum momentum mapping for  $\star$ .*

In the general case of an invariant star product of Wick type, [22, Prop. 4.8] and [22, Cor. 4.10] also transfer directly:

**Proposition 4.3** (cf. [22, Prop. 4.8]) *Let  $J$  be a quantum Hamiltonian for a star product of Wick type  $\star$  with Karabegov's characterizing form  $K(\star)$ , then  $\lambda \in C^2(\mathfrak{g}, C^\infty(M))[[\nu]]$  defined by*

$$\lambda(\xi, \eta) := \frac{1}{\nu} (J(\xi) \star J(\eta) - J(\eta) \star J(\xi)) - J([\xi, \eta]) \quad (14)$$

*lies in  $C^2(\mathfrak{g}, \mathbb{C})[[\nu]]$  and is an element of  $Z_0^2(\mathfrak{g}, \mathbb{C})[[\nu]]$  which is explicitly given by*

$$\lambda(\xi, \eta) = K(\star)(X_\xi, X_\eta) - J([\xi, \eta]) \quad (15)$$

and the cohomology class  $[\lambda] \in H_0^2(\mathfrak{g}, \mathbb{C})[[\nu]]$  does not depend on the choice of  $J$ . Moreover quantum momentum mappings exist if and only if  $[\lambda] = [0] \in H_0^2(\mathfrak{g}, \mathbb{C})[[\nu]]$  and for every  $a \in C^1(\mathfrak{g}, \mathbb{C})[[\nu]]$  such that  $\delta_0 a = \lambda$  the element  $J^a := J - a \in C^1(\mathfrak{g}, C^\infty(M))[[\nu]]$  is a quantum momentum mapping for  $\star$ . Finally, the quantum momentum mapping (if it exists) is unique up to elements in  $Z_0^1(\mathfrak{g}, \mathbb{C})[[\nu]]$ , and hence we have uniqueness if and only if  $H_0^1(\mathfrak{g}, \mathbb{C}) = 0$ .

**Remark 4.4** As for the classical momentum mapping  $J_0$  a quantum Hamiltonian  $J$  for a Lie group action  $r$  that satisfies  $r(g)J(\xi) = J(\text{Ad}(g)\xi)$  for all  $g \in G$  and for all  $\xi \in \mathfrak{g}$  is a quantum momentum mapping and hence  $\text{Ad}^*$ -equivariance of a quantum Hamiltonian is sufficient for it to define a quantum momentum mapping.

We would like to recall the example of semi-simple Lie algebras we gave in [22, Ex. 4.9]. In this case, owing to the Whitehead Lemmas and a deformed version of Sternberg's Lemma, there is a unique quantum momentum mapping for every invariant star product of Wick type. Another important example clearly is that of Abelian Lie algebras, where, even if a quantum momentum mapping exists, it is never unique. Finally, in the case of star products of Wick type we also find that the existence of a classical momentum mapping does not imply the existence of a quantum momentum mapping:

**Corollary 4.5** (cf. [22, Cor. 4.10]) *Let  $\star$  be an invariant star product of Wick type with Karabegov's characterizing form  $K(\star)$ , and assume that there is a classical momentum mapping  $J_0$  for the action  $\varrho$ , then a quantum momentum mapping  $J$  exists if and only if there is an element  $J_+ \in \nu C^1(\mathfrak{g}, C^\infty(M))[[\nu]]$  such that*

$$i_{X_\xi}(K(\star) - \omega) = dJ_+(\xi) \quad \text{and} \quad (K(\star) - \omega)(X_\xi, X_\eta) = (\delta_\varrho J_+)(\xi, \eta) \quad \forall \xi, \eta \in \mathfrak{g}, \quad (16)$$

and these equations determine  $J_+$  up to elements of  $\nu Z_0^1(\mathfrak{g}, \mathbb{C})[[\nu]]$ .

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