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ABSTRACT

The set of functions in C(T) which have a strongly unique best approximation from a given finite-dimensional subspace is denoted by SU(G). Since strong unicity plays an important role in numerical computations and since there the functions are only known up to some error, it is natural to ask what are the functions from the interior of SU(G). A complete characterization of those functions is given and the result is applied to weak Chebyshev and spline subspaces.

O. INTRODUCTION. In this paper we investigate some questions concerning strong unicity of best approximations which arise from numerical considerations. Let G be a finite-dimensional subspace of $C(T)$, the space of all continuous real-valued functions on a compact metric space T endowed with the norm $\| f \| = \sup\{|f(t)| : t \in T\}$. A function $g_f \in G$ is called strongly unique best approximation of $f \in C(T)$ from G, if there exists a constant $K > 0$ such that for all $g \in G$, $\| f - g \| \geq \| f - g_f \| + K \| g - g_f \|$. The strong unicity constant $K(f)$ of f is defined to be the maximum of all such constants K. By $SU(G)$ we denote the set $\{f \in C(T) : f$ has a strongly unique best approximation from $G\}$.

Strong unicity plays an important role in the numerical computation of best approximations. For example the strong unicity constant can be used to estimate the error of a computed approximation with respect to the best approximation. For a given function $f \in C(T)$ Remez type algorithms compute an approximation $g \in G$ of f and a corresponding $\lambda_g > 0$ with the property that

$$\lambda_g \leq d(f,G) \leq \| f - g \| .$$

(see e.g. Remez [15] for polynomials and Nürnberger & Sommer [13] for spline functions). If f has a strongly unique best approximation $g_f \in G$, then we obtain the following estimation:

$$\| g - g_f \| \leq \frac{1}{K(f)} \left(\| f-g \| - \| f-g_f \| \right) \leq \frac{1}{K(f)} \left(\| f-g \| - \lambda_g \right) .$$

Formulas for computing the strong unicity constant have been given by several authors (see e.g. the references in [10] and [11]).

Therefore it is of interest to characterize the functions from $SU(G)$ (see Bartelt & McLaughlin [1], Brosowski [2], Nürnberger, Schumaker, Sommer & Strauß [14], Wulbert [22] and [8], [9]). Since in numerical computation the functions $f \in C(T)$ are only known up to some error, it is natural to ask which functions $f \in SU(G)$ are "stable under small perturbations", i.e. to characterize the functions f from the interior of $SU(G)$.

In section 1 we give a complete characterization of functions $f \in \operatorname{int} SU(G)$ for arbitrary finite-dimensional subspaces G of $C(T)$ by using properties of the error function. Then we apply this result to weak Chebyshev subspaces and spline subspaces in section 2 and 3. It is also shown that those finite-dimensional subspaces G of $C(T)$ for which the set $SU(G)$ is non-empty and open are exactly the Haar subspaces.

Moreover, we consider in particular the role of strong unicity in the computation of best spline approximations.

The results in this paper for $C(T)$ also hold for $C_o(T)$, the space of all continuous real-valued functions on a locally compact metric space T vanishing at infinity.

In a further paper we consider similar questions for semi-infinite optimization problems.

1. FINITE-DIMENSIONAL SUBSPACES. Let $G=\text{span}\{g_1,\dots,g_m\}$ be an m-dimensional subspace of $C(T)$, $f \in C(T)$ and $g_f \in G$. The function g_f is called a best approximation of f from G, if for all $g \in G$, $\| f-g \| \geq \| f-g_f \|$. Given points $t_1,\dots,t_m$ in T we set

$$D\begin{pmatrix} g_1,\dots,g_m \\ t_1,\dots,t_m \end{pmatrix} = \begin{vmatrix} g_1(t_1) \dots g_1(t_m) \\ \vdots \qquad \vdots \\ g_m(t_1) \dots g_m(t_m) \end{vmatrix} .$$

Furthermore, we denote by $E(f)=\{t \in T : |f(t)|=\| f \|\}$ the set of extreme points of f. A closed subset E of $E(f)$ is called extremal (with respect to G), if for all $g \in G$, $\min\{f(t)g(t) : t \in E\} \leq 0$. An extremal subset E of $E(f)$ is said to be primitive, if no closed subset F of E with $F \neq E$ is extremal.

The following result is well-known (see e.g. Singer [19]).

(I) g_f is a best approximation of f from G if and only if for some $p \in \{1,\dots,m+1\}$ there exists an extremal subset $\{t_1,\dots,t_p\}$ of $E(f-g_f)$ if and only if $E(f-g_f)$ is an extremal set.

Moreover, in this section we need the following two theorems on strong unicity.

(II) g_f is a strongly unique best approximation of f from G if and only if for all $g \in G$,

$$\min\{(f(t) - g_f(t))g(t) : t \in E(f - g_f)\} < 0 \quad .$$

(III) g_f is a strongly unique best approximation of f from G if and only if there exists a subset $\{t_1, \ldots, t_m\}$ of $\cup\{E : E \text{ is a primitive extremal subset of } E(f - g_f)\}$ with

$$D\begin{pmatrix} g_1, \ldots, g_m \\ t_1, \ldots, t_m \end{pmatrix} \neq 0 \quad .$$

Statement (II) follows from a result of Wulbert [22] on normed linear spaces and statement (III) is due to Brosowski [2].

Since in numerical computation the functions $f \in C(T)$ are only given up to some error, it is natural to ask which functions $f \in SU(G)$ are "stable under small perturbations", i.e. which are the functions f from $\operatorname{int} SU(G)$. By $\operatorname{int} SU(G)$ we denote the interior of $SU(G)$. We give the following characterization. Note that, if T is a finite set, then by Nürnberger & Singer [12] $SU(G) = U(G) = \{f \in C(T) : f \text{ has a unique best approximation from } G\}$.

THEOREM 1.1. Let $G = \operatorname{span}\{g_1, \ldots, g_m\}$ be an m-dimensional subspace of $C(T)$, $f \in C(T) \setminus G$ and $g_f \in G$ be a best approximation of f. Then the following statements (1) and (2) are equivalent:

(1) $f \in \operatorname{int} SU(G)$

(2)(a) $f - g_f$ has at least $m+1$ extreme points.

(b) For every subset $\{t_1,\ldots,t_{m+1}\}$ of $E(f-g_f)$ which consists of $m+1$ distinct points and is extremal with respect to G,

$$D\begin{pmatrix} g_1, & \ldots\ldots & ,g_m \\ t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{m+1} \end{pmatrix} \neq 0 \quad , \quad i=1,\ldots,m+1 \quad .$$

<u>Proof</u>. We first show that (1) $\Rightarrow$ (2a).

Let $f \in \operatorname{int} SU(G)$. It follows from (III) that there exists a subset $\{t_1,\ldots,t_m\}$ of $E(f-g_f)$ with

$$D\begin{pmatrix} g_1,\ldots,g_m \\ t_1,\ldots,t_m \end{pmatrix} \neq 0 \quad .$$

If $E(f-g_f)=\{t_1,\ldots,t_m\}$, then by interpolation $E(f-g_f)$ is not an extremal set contradicting statement (I).

Now we show that (1) $\Rightarrow$ (2b).

Assume that (2b) fails, i.e. there exists an extremal subset $M=\{t_1,\ldots,t_{m+1}\}$ of $E(f-g_f)$ such

$$D\begin{pmatrix} g_1, & \ldots\ldots & ,g_m \\ t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{m+1} \end{pmatrix} = 0 \quad \text{for some } j\in\{1,\ldots,m+1\}.$$

We set $h=f-g_f$ and show that

(3) there exists a sequence (h_n) in $C(T)$ such that $h_n \to h$, $h_n(t)=h(t)$ for all $t\in M$ and $E(h_n)=M$ for all n.

For the moment we assume that (3) holds. Since M is an extremal subset of $E(h)$ and $h_n(t)=h(t)$, $t\in M$, M is an extremal subset of $E(h_n)$, it follows from (I) that zero is a best approximation of h_n. Moreover, since

$$D\begin{pmatrix} g_1, & \ldots\ldots & ,g_m \\ t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{m+1} \end{pmatrix} = 0 \quad ,$$

there exists a function $g\in G$, $g\neq 0$, such that

$g(t_i) = 0$, $i=1,\dots,m+1$, $i \neq j$, and $h_n(t_j) \cdot g(t_j) \geq 0$. Thus by (II) zero is not a strongly unique best approximation of h_n .

We set $f_n = h_n + g_f$ for all n . Then $f_n \to f$ and for all n , g_f is a best approximation of f_n , but not a strongly unique one. This shows that $f \notin \operatorname{int} SU(G)$. Therefore it remains to prove (3).

Let (V_n) be an open neighborhood basis of M . Since T is a compact metric space, it follows from Urysohn's lemma that for each n there exists a function $z_n \in C(T)$ such that $z_n(t) = 1$, $t \in M$, $z_n(t) = 1 - \frac{1}{n}$, $t \in T \setminus V_n$ and $1 - \frac{1}{n} \leq z_n(t) < 1$ for all $t \in T \setminus \{t_1,\dots,t_{m+1}\}$.

We set for all n , $h_n = z_n \cdot h$. Then (h_n) has the desired property. This shows that (1) $\Rightarrow$ (2).

Now we show that (2) $\Rightarrow$ (1).

We assume that (2) holds. Since g_f is a best approximation of f , it follows from (I) and (2a) that there exists an extremal subset $\{t_1,\dots,t_{m+1}\}$ of $E(f-g_f)$ consisting of $m+1$ distinct points. By (2b) the set $\{t_1,\dots,t_{m+1}\}$ is a primitive extremal subset of $E(f-g_f)$, since we can interpolate at any n points of $\{t_1,\dots,t_{m+1}\}$. Thus it follows from (III) that $f \in SU(G)$.

Now we assume that (1) fails, i.e. there exists a sequence (f_n) in $C(T)$ such that $f_n \to f$ and f_n does not have a strongly unique best approximation for all n . For each n we choose a best approximation $g_n \in G$ of f_n . It follows from (I) that for each n there exists an integer $p_n \in \{1,\dots,m+1\}$ and a subset $M_n = \{t_{1,n},\dots,t_{p_n,n}\}$ of $E(f_n-g_n)$ such that for all $g \in G$,

(4) $\min\{(f_n(t)-g_n(t))g(t) : t \in M_n\} \leq 0$.

Going to a subsequence of (f_n) we may assume that for all n, $p_n = p \in \{1,\ldots,m+1\}$, $g_n \to g_f$ and $t_{i,n} \to t_i \in T$, $i=1,\ldots,p$.

Since for all n, $|f_n(t_{i,n}) - g_n(t_{i,n})| = \| f_n - g_n \|$, $i=1,\ldots,p$, it follows by taking limits that

$$|f(t_i) - g_f(t_i)| = \| f - g_f \| \ , \ i=1,\ldots,p \ ,$$

which implies that $M = \{t_1,\ldots,t_p\}$ is a subset of $E(f-g_f)$.

Let $g \in G$ be given. It follows from (4) that for each n there exists an integer $j_n \in \{1,\ldots,p\}$ such that $\left(f_n(t_{j_n,n}) - g_n(t_{j_n,n})\right) g(t_{j_n,n}) \leq 0$.

Going to a subsequence and taking limits it follows that there exists a $j \in \{1,\ldots,p\}$ such that $\left(f(t_j) - g_f(t_j)\right) g(t_j) \leq 0$.

This shows that M is an extremal subset of $E(f-g_f)$. We show that this leads to a contradiction. By omitting some points of M, if necessary, we may assume that all points of $M = \{t_1,\ldots,t_p\}$ are distinct.

<u>CASE 1</u>. $p \in \{1,\ldots,m\}$.

By (2a) we can choose points $t_{p+1},\ldots,t_{m+1}$ in $E(f-g_f)$ such that all points of $\{t_1,\ldots,t_{m+1}\}$ are distinct. Since $\{t_1,\ldots,t_{m+1}\}$ is an extremal subset of $E(f-g_f)$, it follows from (2b) that

$$D\begin{pmatrix} g_1,\ldots,g_m \\ t_1,\ldots,t_m \end{pmatrix} \neq 0 \ .$$

Therefore we can interpolate at $t_1,\ldots,t_m$ which contradicts the fact that $\{t_1,\ldots,t_p\}$ is an extremal subset of $E(f-g_f)$.

<u>CASE 2</u>. $p = m+1$.

In this case $M_n = \{t_{1,n},\ldots,t_{m+1,n}\}$ for all n. Assume that there exists an integer n such that for all $i \in \{1,\ldots,m+1\}$,

$$D\begin{pmatrix} g_1 & , & \cdot & \cdot & \cdot & \cdot & \cdot & ,g_m \\ t_{1,n}, & \ldots, & t_{i-1,n}, & t_{i+1,n}, & \ldots, & t_{m+1,n} \end{pmatrix} \neq 0 .$$

Then as above M_n is a primitive extremal subset of $E(f_n-g_n)$ and by (III), $f_n \in SU(G)$, a contradiction. Thus going to a subsequence we may assume that there exists an integer $j \in \{1,\ldots,m+1\}$ such that for all n,

$$D\begin{pmatrix} g_1 & , & \cdot & \cdot & \cdot & \cdot & \cdot & ,g_m \\ t_{1,n}, & \ldots, & t_{j-1,n}, & t_{j+1,n}, & \ldots, & t_{m+1,n} \end{pmatrix} = 0 .$$

By taking limits it follows that

$$D\begin{pmatrix} g_1, & \cdot & \cdot & \cdot & \cdot & \cdot & ,g_m \\ t_1, & \ldots, & t_{j-1}, & t_{j+1}, & \ldots, & t_{m+1} \end{pmatrix} = 0 .$$

Since $\{t_1,\ldots,t_{m+1}\}$ is an extremal subset of $E(f-g_f)$ thus contradicts (2b).

This shows (2) $\Rightarrow$ (1) and completes the proof of Theorem 1.1.

<u>COROLLARY 1.2.</u> Let G be an m-dimensional subspace of $C(T)$, $f \in C(T) \setminus G$ and $g_f \in G$ such that $f - g_f$ has exactly $m+1$ extreme points. Then $f \in SU(G)$ if and only if $f \in \text{int } SU(G)$.

<u>Proof.</u> Let $E(f-g_f) = \{t_1,\ldots,t_{m+1}\}$ and $f \in SU(G)$. Assume that there exists an integer $j \in \{1,\ldots,m+1\}$ such that

$$D\begin{pmatrix} g_1, & \cdot & \cdot & \cdot & \cdot & \cdot & ,g_m \\ t_1, & \ldots, & t_{j-1}, & t_{j+1}, & \ldots, & t_{m+1} \end{pmatrix} = 0 .$$

Then there exists a function $g \in G$, $g \neq 0$, such that $g(t_i) = 0$, $i=1,\ldots,m+1$, $i \neq j$, and $(f(t_j) - g_f(t_j))g(t_j) \geq 0$. This contradicts (II). Now, it follows from Theorem 1.1 that $f \in \text{int } SU(G)$. This shows Corollary 1.2.

COROLLARY 1.3. Let $G = \text{span}\{g_1, \ldots, g_m\}$ be an m-dimensional subspace of $C(T)$, $f \in C(T) \setminus G$ and $g_f \in G$ such that $E(f-g_f) = \{t_1, \ldots, t_{m+1}\}$ is a set of $m+1$ distinct points. Then the following statements are equivalent:

(1) $f \in \text{int } SU(G)$ and g_f is the strongly unique best approximation of f from G.

(2) There exists an integer $\sigma \in \{-1,1\}$ such that

$$\sigma \cdot \sigma_i (-1)^i (f(t_i) - g_f(t_i)) = \| f - g_f \| \ , \ i=1,\ldots,m+1,$$

where

$$\sigma_i = D\begin{pmatrix} g_1, & & & & , g_m \\ t_1, \ldots, & t_{i-1}, & t_{i+1}, & \ldots, & t_{m+1} \end{pmatrix} \neq 0 \ , \ i=1,\ldots,m+1,$$

Proof. Since $E(f-g_f) = \{t_1, \ldots, t_{m+1}\}$, it follows from (I) and Theorem 1.4 on p.182 in Singer [19] that $\{t_1, \ldots, t_{m+1}\}$ is an extremal subset if and only if there exists an integer $\sigma \in \{-1,1\}$ such that

$$\sigma \cdot \sigma_i (-1)^i (f(t_i) - g_f(t_i)) = \| f - g_f \| \ , \ i=1,\ldots,m+1.$$

Using this fact Corollary 1.3 follows immediately from Theorem 1.1 and (I).

An m-dimensional subspace G of $C(T)$ is called Haar subspace, if every $g \in G$, $g \neq 0$, has at least $m-1$ distinct zeros.

The next result shows that for non Haar subspace G of $C(T)$ the set $SU(G)$ is not open or empty.

COROLLARY 1.4. Let T have no isolated points and G be a finite-dimensional subspace of $C(T)$. Then the following statements are equivalent:

(1) $SU(G)$ is a non-empty open subset of $C(T)$.

(2) G is a Haar subspace.

Proof. If G is a Haar subspace, then it follows from Newman & Shapiro [7] that $SU(G) = C(T)$, i.e. $SU(G)$ is a non-empty open set.

Now we prove the converse. We assume that (1) holds and (2) fails. It follows from (1) that there exists a function $f \in \operatorname{int} SU(G)$. Let $g_f \in G$ be the strongly unique best approximation of f . Then by (I) and Theorem 1.1 there exists a subset $\{u_1, \ldots, u_{m+1}\}$ of $E(f-g_f)$ such that

$$(3) \qquad D\begin{pmatrix} g_1, & \ldots & , g_m \\ u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{m+1} \end{pmatrix} \neq 0 , \quad i=1, \ldots, m+1 ,$$

where $G = \operatorname{span}\{g_1, \ldots, g_m\}$.

Since G is not a Haar subspace there exists a subset $\{t_1, \ldots, t_m\}$ of T such that

$$D\begin{pmatrix} g_1, \ldots, g_m \\ t_1, \ldots, t_m \end{pmatrix} = 0 .$$

It follows from (3) that for each $i \in \{1, \ldots, m+1\}$ there exists a neighbourhood V_i of t_i such that for all distinct integers $q_1, \ldots, q_m \in \{1, \ldots, m+1\}$ and all points $v_i \in V_{q_i}$, $i=1, \ldots, m$,

$$D\begin{pmatrix} g_1, \ldots, g_m \\ v_1, \ldots, v_m \end{pmatrix} \neq 0 .$$

Since T has no isolated points, for each $i \in \{1, \ldots, m+1\}$ we can choose a point $w_i \in V_i$ such that $\{w_1, \ldots, w_{m+1}\} \cap \{t_1, \ldots, t_m\} = \emptyset$. Now, we construct a function $f \in C(T)$ such that $f \in SU(G) \setminus \operatorname{int} SU(G)$.

Since

$$D\begin{pmatrix} g_1, \ldots, g_m \\ t_1, \ldots, t_m \end{pmatrix} = 0 ,$$

there exist real numbers $\alpha_1, \ldots, \alpha_m$ such that

$\sum_{i=1}^{m} |\alpha_i| \neq 0$ and for each $g \in G$, $\sum_{i=1}^{m} \alpha_i g(t_i) = 0$.

We define f on $\{w_1,\dots,w_{m+1}\} \cup \{t_1,\dots,t_m\}$ as follows:

(4) $f(t_i) = \operatorname{sgn} \alpha_i$, if $\alpha_i \neq 0$, and
$f(t_i) = 1$, if $\alpha_i = 0$, $i=1,\dots,m$.

(5) $f(w_i) = \sigma_i (-1)^i$, $i=1,\dots,m+1$,

where

(6) $\sigma_i = D\begin{pmatrix} g_1, & \dots\dots\dots & ,g_m \\ w_1,\dots,w_{i-1}, & w_{i+1},\dots, & w_{m+1} \end{pmatrix} \neq 0$, $i=1,\dots,m+1$.

By Tietze's extension theorem we can extend f continuously to T such that $\|f\| = 1$.

By using (5) and (6) it follows from (I) and Theorem 1.3 on p. 178 in Singer [19] that $\{w_1,\dots,w_{m+1}\}$ is a primitive extremal subset of $E(f)$. Thus by (6) and (III) zero is a strongly unique best approximation of f, i.e. $f \in SU(G)$.

Moreover, assume that there exists a $g \in G$ such that for all $i \in \{1,\dots,m\}$, $f(t_i)g(t_i) > 0$. Then
$\sum_{i=1}^{m} \alpha_i g(t_i) = \sum_{i=1}^{m} |\alpha_i| f(t_i) g(t_i) > 0$, a contradiction.

This shows that $\{t_1,\dots,t_m\}$ is an extremal subset of $E(f)$. Then also $\{t_1,\dots,t_m,w_1\}$ is an extremal subset of $E(f)$. But since

$$D\begin{pmatrix} g_1,\dots,g_m \\ t_1,\dots,t_m \end{pmatrix} = 0$$

it follows from Theorem 1.1 that $f \notin \operatorname{int} SU(G)$. Therefore, the set $SU(G)$ is not open, which contradicts (1). This shows Corollary 1.4.

REMARK 1.5. Let us consider extensions of Corollary 1.4. We assume that T is an arbitrary compact metric space and that $G = \operatorname{span}\{g_1,\dots,g_m\}$ is a finite-dimensional subspace of $C(T)$. If G is a Haar subspace, then by the proof of Corollary 1.4 $SU(G) = C(T)$ is a non-empty open set.

Conversely, if G is not a Haar subspace, then there exists a subset $M_1 = \{t_1,\dots,t_m\}$ of T such that

$$D\begin{pmatrix} g_1,\dots,g_m \\ t_1,\dots,t_m \end{pmatrix} = 0$$

and in realistic approximation problems (also if T is a finite set) there exists a further subset $M_2 = \{w_1,\dots,w_{m+1}\}$ of T such that $M_1 \cap M_2 = \emptyset$ and

$$D\begin{pmatrix} g_1, & \dots\dots\dots & ,g_m \\ w_1,\dots,w_{i-1},w_{i+1},\dots,w_{m+1} \end{pmatrix} \neq 0 \ , \ i=1,\dots,m+1 \ .$$

Then by the proof of Corollary 1.4 $SU(G)$ is not a non-empty open set.

REMARK 1.6. The results applied in the proof of Theorem 1.1 also hold for normed linear spaces (see Brosowski [2], Singer [19] and Wulbert [22]). Thus the proof can be used to show that the implication (2) ⇒ (1) in Theorem 1.1 has an obvious analog for normed linear spaces (compare the notation on p. 245 in Brosowski [2]).

2. WEAK CHEBYSHEV SUBSPACES.

Now we apply the results in section 1 to weak Chebyshev subspaces.

Throughout this section let T be a compact subset of the real line. An m-dimensional subspace G of $C(T)$ is called weak Chebyshev, if all functions $g \in G$ have at most $m-1$ sign changes, i.e. there do not exist points $t_1 < \dots < t_{m+1}$ in T with $g(t_i)\cdot g(t_{i+1}) < 0$, $i=1,\dots,m$. We call points $t_1 < \dots < t_p$ in T alternating extreme points of a function $f \in C(T)$, if there exists an integer $\sigma \in \{-1,1\}$ such that $\sigma(-1)^i f(t_i) = \| f \|$, $i=1,\dots,p$.

The following characterization is due to Jones & Karlovitz [5] for $T = [a,b]$ and in the final form due to Deutsch, Nürnberger & Singer [4].

(IV) An m-dimensional subspace G of $C(T)$ is weak Chebyshev if and only if for each $f \in C(T)$ there exists a best approximation $g_f \in G$ such that $f - g_f$ has at least $m+1$ alternating extreme points.

By using Theorem 1.1 we prove the following characterization of functions $f \in \text{int } SU(G)$ for weak Chebyshev subspaces G.

THEOREM 2.1. Let $G = \text{span}\,\{g_1,\ldots,g_m\}$ be an m-dimensional weak Chebyshev subspace of $C(T)$, $f \in C(T) \setminus G$ and $g_f \in G$ be a best approximation of f. Then the following statements (1) and (2) are equivalent.

(1) $f \in \text{int } SU(G)$.

(2)(a) $f - g_f$ has at least $m+1$ alternating extreme points.

(b) For every set $\{t_1,\ldots,t_{m+1}\}$ of $m+1$ alternating extreme points of $f - g_f$

$$D\begin{pmatrix} g_1 & \ldots\ldots\ldots & ,g_m \\ t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{m+1} \end{pmatrix} \neq 0\ ,\quad i=1,\ldots,m+1.$$

Proof. We first show that (1) ⇒ (2).
It follows from (IV) that (1) ⇒ (2a).
Let $\{t_1,\ldots,t_{m+1}\}$ be a set of $m+1$ alternating extreme points of $f - g_f$. If there exists a $g \in G$, $g \neq 0$, such that for all $i \in \{1,\ldots,m+1\}$,

$$(f(t_i) - g_f(t_i))g(t_i) > 0\ ,$$

then g has at least m sign changes, which contradicts that G is weak Chebyshev. Therefore

$\{t_1,\dots,t_{m+1}\}$ is an extremal subset of $E(f-g_f)$ and the implication (1) $\Rightarrow$ (2b) follows from Theorem 1.1. Now we show that (2) $\Rightarrow$ (1). We assume that (2) holds. It follows from Corollary 1.6 in [9] (see also [8]) that $f \in SU(G)$. Now we assume that (1) fails, i.e. there exists a sequence (f_n) in $C(T)$ such that $f_n \to f$ and f_n does not have a strongly unique best approximation for all n. It follows from (IV) that for each n there exists a best approximation $g_n \in G$ such that $f_n - g_n$ has at least $m+1$ alternating extreme points, i.e. there exist $t_{1,n} < \dots < t_{m+1,n}$ in T and a $\sigma_n \in \{-1,1\}$ with

$$\sigma_n(-1)^i(f_n(t_{i,n}) - g_n(t_{i,n})) = \| f_n - g_n \| , \; i=1,\dots,m+1.$$

Since for all n, g_n is not a strongly unique best approximation of f_n, it follows from Corollary 1.6 in [9] that for each n there exists an integer $j_n \in \{1,\dots,m+1\}$ such that

$$D\begin{pmatrix} g_1, & \dots & \dots & \dots & ,g_m \\ t_{1,n},\dots,t_{j_n-1,n}, & t_{j_n+1,n},\dots,t_{m+1,n} \end{pmatrix} = 0$$

Going to a subsequence we may assume that for all n, $\sigma_n = \sigma \in \{-1,1\}$, $j_n = j \in \{1,\dots,m+1\}$, $g_n \to g_f$ and $t_{i,n} \to t_i \in T$, $i=1,\dots,m+1$. Taking limits it follows that

$$\sigma(-1)^i(f(t_i) - g_f(t_i)) = \| f - g_f \| , \; i=1,\dots,m+1 ,$$

and

$$D\begin{pmatrix} g_1, & \dots & \dots & ,g_m \\ t_1,\dots,t_{j-1}, & t_{j+1},\dots,t_{m+1} \end{pmatrix} = 0 ,$$

which contradicts (2). This shows (2) $\Rightarrow$ (1) and completes the proof of Theorem 2.1.

<u>COROLLARY 2.2.</u> Let $G = \mathrm{span}\{g_1,\dots,g_m\}$ be an m-dimensional weak Chebyshev subspace of $C(T)$,

$f \in C(T) \setminus G$ and $g_f \in G$ such that

$$E(f-g_f) = \{t_1,\dots,t_{m+1}\}$$

and $t_1 < \dots < t_{m+1}$. Then the following statements are equivalent:

(1) $f \in \operatorname{int} SU(G)$ and g_f is the strongly unique best approximation of f from G .

(2) The points $t_1 < \dots < t_{m+1}$ are alternating extreme points of $f-g_f$ and

$$D\begin{pmatrix} g_1, & \dots\dots\dots & ,g_m \\ t_1,\dots,t_{i-1},t_{i+1},\dots,t_{m+1} \end{pmatrix} \neq 0 \ , \ i=1,\dots,m+1.$$

<u>Proof.</u> It is well-known that if $f-g_f$ has at least $m+1$ alternating extreme points, then g_f is a best approximation of f from G . Using this fact Corollary 2.2 follows immediately from Theorem 2.1.

<u>3. SPLINE SUBSPACES.</u> Now we apply the results in section 2 to the prototypes of weak Chebyshev subspaces, namely the spline subspaces.

Let k fixed knots $a = x_0 < x_1 < \dots < x_k < x_{k+1} = b$ be given. The subspace of <u>polynomial spline functions</u> of degree n with k fixed $x_1,\dots,x_k$ is defined by

$$S_n(x_1,\dots,x_k) = \{s \in C^{(n-1)}[a,b] : s\big|_{[x_i,x_{i+1}]} \text{ is a polynomial of degree } n \ , \ i=0,1,\dots,k\} \ .$$

Let $\{\tilde{s}_1,\dots,\tilde{s}_{n+k+1}\}$ be a basis of $S_n(x_1,\dots,x_k)$.

The first result on interpolation is due to Schoenberg & Whitney (see Schumaker [18]).

(V) Let points $a \le t_1 < \dots < t_{n+k+1} \le b$ be given. Then

$$D\begin{pmatrix} \tilde{s}_1,\dots,\tilde{s}_{n+k+1} \\ t_1,\dots,t_{n+k+1} \end{pmatrix} \neq 0$$

if and only if each interval $[x_o,x_j)$ and $(x_{k+1-j},x_{k+1}]$, $j=1,\dots,k+1$, contains at least j points from $\{t_1,\dots,t_{n+k+1}\}$.

A characterization of best spline approximations has been given by Rice [16] and Schumaker [17].

(VI) A function $s_f \in S_n(x_1,\dots,x_k)$ is a best approximation of $f \in C[a,b]$ if and only if there exists an interval $[x_p,x_{p+q}]$ such that $f-s_f$ has at least $n+q+1$ alternating extreme points in $[x_p,x_{p+q}]$.

The next result is a characterization of strongly unique best spline approximations which follows from a more general result on weak Chebyshev subspaces both given in [8],[9].

(VII) A function $s_f \in S_n(x_1,\dots,x_k)$ is a strongly unique best approximation of $f \in C[a,b] \setminus S_n(x_1,\dots,x_k)$ if and only if $f-s_f$ has at least $n+k+2$ alternating extreme points in $[a,b]$ and at least $j+1$ alternating extreme points in each interval $[x_o,x_j)$, $(x_{k+1-j},x_{k+1}]$, $(x_i,x_{i+j+n}) \subset [a,b]$ $(j \geq 1)$.

By using Theorem 2.1 we prove the following characterization of functions $f \in \operatorname{int} SU(S_n(x_1,\dots,x_k))$ and $f \in \operatorname{int} U(S_n(x_1,\dots,x_k))$.

<u>THEOREM 3.1.</u> Let $f \in C[a,b] \setminus S_n(x_1,\dots,x_k)$ and $s_f \in S_n(x_1,\dots,x_k)$ be a best approximation of f. Then the following statements (1), (2) and (3) are equivalent:

(1) $f \in \operatorname{int} SU(S_n(x_1,\dots,x_k))$

(2) $f \in \operatorname{int} U(S_n(x_1,\dots,x_k))$

(3)(a) $f-s_f$ has at least $n+k+2$ alternating extreme points in $[a,b]$.

(b) $f - s_f$ has at most $n+q$ alternating extreme points in each interval $[x_p, x_{p+q}] \subsetneqq [a,b]$.

<u>Proof</u>. The implication (1) ⇒ (2) is obvious.
Now we show that (3) ⇒ (1). We assume that (3) holds and show that then (2) in Theorem 2.1 is satisfied. Assume that (2b) in Theorem 2.1 fails, i.e. there exists a set of alternating extreme points $\{t_1, \ldots, t_{n+k+2}\}$ of $f - s_f$ such that for some $i \in \{1, \ldots, n+k+2\}$

$$D\begin{pmatrix} \tilde{s}_1, & \ldots\ldots\ldots & ,\tilde{s}_{n+k+1} \\ t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n+k+2} \end{pmatrix} = 0 .$$

It follows from (V) that there exists an interval $[x_o, x_j)$ or $(x_{k+1-j}, x_{k+1}]$ which contains at most $j-1$ points of $\{t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n+k+2}\}$. Then $[x_j, x_{k+1}]$ or $[x_o, x_{k+1-j}]$ contains at least $n+k+2-j$ points of $\{t_1, \ldots, t_{n+k+2}\}$, contradicting (3b). This shows that (3) ⇒ (1).
Now we show that (2) ⇒ (3). Let $f \in \operatorname{int} U(S_n(x_1, \ldots, x_k))$. It follows from (IV) that (3a) is satisfied. Now we assume that (3b) fails, i.e. there exists an interval $[x_p, x_{p+q}] \subsetneqq [a,b]$ which contains at least $n+q+1$ alternating extreme points $x_p \le t_1 < \ldots < t_{n+q+1} \le x_{p+q}$ of $f - s_f$. We set $h = f - s_f$. As in the proof of Theorem 1.1, (1) ⇒ (2) there exists a sequence (h_m) in $C[a,b]$ such that $h_m \to h$, $h_m(t_i) = h(t_i)$, $i=1,\ldots,n+q+1$ and $E(h_m) = \{t_1, \ldots, t_{n+q+1}\}$. Then it follows from (VI) that zero is a best approximation of h_m . Since $[x_o, x_p)$ and $(x_{p+q}, x_{k+1}]$ contain no alternating extreme point of $f - s_f$ and $x_o < x_p$ or $x_{p+q} < x_{k+1}$, it follows from Theorem 2.4 in Nürnberger & Singer [12] that zero is not a unique best approximation of h_n . For all m , we set $f_m = h_m + s_f$. Then $f_m \to f$ and for

all m, s_f is a best approximation of f_m, but not a unique one. This shows that $f \notin \text{int}\, U(S_n(x_1,\ldots,x_k))$, a contradiction.
This shows (2) $\Rightarrow$ (3) and completes the proof of Theorem 3.1.

REMARK 3.2. The connection between condition (3) in Theorem 3.1 and the condition in (VII) is easy to verify. Let us consider the interval $[x_o,x_j)$ in (VII). Since by (3) in Theorem 3.1, $f-s_f$ has at least $n+k+2$ alternating extreme points in $[a,b]$ and at most $n+k+1-j$ alternating extreme points in $[x_j,x_{k+1}]$, it follows that $f-s_f$ has at least $j+1$ alternating extreme points in $[x_o,x_j)$. The arguments for the other intervals in (VII) are similar.

The next result is an immediate consequence of Theorem 3.1.

COROLLARY 3.3. No function in

$$U(S_n(x_1,\ldots,x_k)) \setminus SU(S_n(x_1,\ldots,x_k))$$

is an interior point of $U(S_n(x_1,\ldots,x_k))$.

REMARK 3.4. Strong unicity plays a special role in the computation of best spline approximations:

A Remez type algorithm for $S_n(x_1,\ldots,x_k)$ was developed in Nürnberger & Sommer [13] and it was shown that for a given function $f \in C[a,b]$ the algorithm converges to a best approximation of f, if $k \le n+1$, and to a nearly best approximation, if $k > n+1$. If f has a strongly unique best approximation s_f, then the algorithm always converges to s_f.

Moreover, when we tested our algorithm, we saw by using (VII) that many standard functions $f \in C[a,b]$ actually have a strongly unique best approximation $s_f \in S_n(x_1,...,x_k)$ and $f - s_f$ has exactly $n+k+2$ extreme points, which implies that $f \in \operatorname{int} SU(S_n(x_1,...,x_k))$ (see Corollary 1.2). In this context compare also Theorem 3.6 below.

Veidinger [20] showed that under certain assumptions the Remez algorithm for Haar subspaces converges quadratically (see also Wetterling [21]). The next theorem shows that an analogous result holds for the simultaneous exchange algorithm in [13], when f has a strongly unique best approximation.

<u>DEFINITION 3.5.</u> Let $f \in C[a,b] \setminus S_n(x_1,...,x_k)$ and $s_f \in S_n(x_1,...,x_k)$ be a strongly unique best approximation of f. The algorithm in [13] yields for the given f a sequence of spline functions (s_m) in $S_n(x_1,...,x_k)$ and a sequence of real numbers (λ_m). We say that the algorithm <u>converges quadratically</u>, if

$$\| s_f - s_{m+1} \| = \mathcal{O}(\| s_f - s_m \|^2)$$

and

$$d(f,S_n(x_1,\dots,x_k)) - |\lambda_{m+1}| = \mathcal{O}\Big((d(f,S_n(x_1,\dots,x_k) - |\lambda_m|)^2 \Big).$$

<u>THEOREM 3.6.</u> Let $n \geq 3$ and $f \in C^{(2)}[a,b]$ have a strongly unique best approximation $s_f \in S_n(x_1,...,x_k)$. If $f - s_f$ has exactly $n+k+2$ extreme points, $(f - s_f)''(t) \neq 0$ for all $t \in E(f - s_f) \cap (a,b)$ and $(f - s_f)'(t) \neq 0$ for all $t \in E(f - s_f) \cap \{a,b\}$, then the simultaneous exchange algorithm in [13] converges quadratically.

Proof. The arguments are similar as for the Remez algorithm for Haar subspaces (see Theorem 84 in Meinardus [6]) by using the following facts. It was shown in [13] that $|\lambda_m| \to d(f,S_n(x_1,\ldots,x_k))$ and $s_m \to s_f$. Since s_f is a strongly unique best approximation of f and $f-s_f$ has exactly $n+k+2$ extreme points $a \le t_1 < \ldots < t_{n+k+2} \le b$, it follows from Corollary 1.8 in [9] that for all $i \in \{1,\ldots,n+k+2\}$

$$D\begin{pmatrix} \tilde{s}_1, & \ldots\ldots & ,\tilde{s}_{n+k+1} \\ t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{n+k+2} \end{pmatrix} \neq 0 .$$

Then it follows from the proof of Theorem 84 in Meinardus [6] that for sufficiently large m the algorithm in [13] coincides with Newton's method and therefore converges quadratically. This shows Theorem 3.6.

REMARK 3.7. The results in this section (except Theorem 3.6) have obvious analogs for generalized spline subspaces which satisfy the interlacing property (see Nürnberger, Schumaker, Sommer & Strauß [14]).

REFERENCES

1. M.W. Bartelt and H.W. McLaughlin, Characterizations of strong unicity in approximation theory, J. Approximation Theory 9 (1973), 255-266.

2. B. Brosowski, A refinement of the Kolmogorov-Criterion, in Constructive Function Theory, Sofia 1981, 241-247.

3. B. Brosowski, Parametric Semi-Infinite Optimization, Verlag Peter Lang, Frankfurt 1982.

4. F. Deutsch, G. Nürnberger and I. Singer, Weak Chebyshev subspaces and alternation, Pacific J. Math. 89 (1980), 9-31.

5. R.C. Jones and L.A. Karlovitz, Equioscillation under nununiqueness in the approximation of continuous functions, J.Approximation Theory 3 (1970), 138-145.

6. G. Meinardus, Approximation of Functions: Theory and Numerical Methods, Springer-Verlag,Berlin 1967.

7. D.J. Newman and H.S. Shapiro, Some theorems on Chebyshev approximation, Duke Math. J. 30 (1963), 673-684.

8. G. Nürnberger, Strong uniqueness of best approximations and weak Chebyshev systems, in Quantitative Approximation, Bonn 1979, R. DeVore and K. Scherer ed., Academic Press, 1980, 255-266.

9. G. Nürnberger, A local version of Haar's theorem in approximation theory, Numer. Funct. Anal. and Optimiz. 5 (1982), 21-46.

10. G. Nürnberger, Strong unicitiy constants for spline functions, Numer. Funct. Anal. and Optimiz. 5 (1982-83), 319-347.

11. G. Nürnberger, Strong unicity constants for finite-dimensional subspaces, to appear in Approximation Theory IV, College Station 1983, L.L. Schumaker ed., Academic Press.

12. G. Nürnberger and I. Singer, Uniqueness and strong uniqueness of best approximations by spline subspaces and other subspaces, J. Math. Anal. Appl. 90 (1982), 171-184.

13. G. Nürnberger and M. Sommer, A Remez type algorithm for spline functions, Numer. Math. 41 (1983), 117-146.

14. G. Nürnberger, L.L. Schumaker, M. Sommer and H. Strauß, Approximation by generalized splines, preprint.

15. E. Remez, Sur la dêtermination des polynômes d'approximation de degré donnée, Comm. Soc. Math. Kharkov 10 (1934), 41-63.

16. J.C. Rice, Characterization of Chebyshev approximation by splines, SIAM J. Numer. Anal. 4 (1967), 557-565.

17. L.L.Schumaker, Uniform approximation by Tchebycheffian spline functions, J. Math. Mech. 18 (1968), 369-378.

18. L.L. Schumaker, Spline Functions: Basic Theory, Wiley-Interscience, New York 1981.

19. I. Singer, Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer-Verlag, Berlin 1970.

20. L. Veidinger, On the numerical determination of the best approximations in the Chebyshev sense,

Numer. Math. 2 (1960), 99-105.

21. W. Wetterling, Anwendung des Newtonschen Iterationsverfahrens bei der Tschebyscheff- Approximation, insbesondere mit nicht-linear auftretenden Parametern, MTW (1963), Teil I: 61-63, Teil II: 112-115.

22. D.E. Wulbert, Uniqueness and differential characterization of approximations from manifolds of functions, Amer. J. Math. 18 (1971), 350-366.