Nr.74 - 1987

The description of an $\mathbb{R}^n$-valued one form relative to an embedding

E.Binz

(Universität Mannheim)

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1) Introduction

Let M be a compact, smooth and oriented manifold. The collection of all smooth embeddings of M into $\mathbb{R}^n$ equipped with the C^∞-topology is a Frêchet manifold and is denoted by $E(M,\mathbb{R}^n)$. The collection $A^1(M,\mathbb{R}^n)$ consisting of all smooth $\mathbb{R}^n$-valued one forms of M , also endowed with the C^∞ topology, is a Frêchet space. $\mathbb{R}^n$ is assumed to be oriented and is equipped with a fixed scalar product $<,>$.

These notes describe relations between a given $\alpha \in A^1(M,\mathbb{R}^n)$ and a given $i \in E(M,\mathbb{R}^n)$. Two sorts of decomposition of α relative to i are discussed and related. First we show that α splits into

$$\alpha = dh + \beta \qquad h \in C^\infty(M,\mathbb{R}^n) ,$$

where β has only the zero as integrated parts. The other decomposition is

$$\alpha = c_\alpha(i)\cdot di + di\cdot C_\alpha(i) + di\cdot B_\alpha(i)$$

with $c_\alpha \in C^\infty(M,so(n))$ and $C_\alpha(i)$ as well as $B_\alpha(i)$ are smooth, strong bundle maps of TM skew-, respectively selfadjoint with respect to the Riemannian metric $m(i)$, the pullback of $< >$ by i .

The relation among these two splittings as well as the techniques presented play a crucial role in elasticity theory.

Smoothness is always meant in the sense of [Gu] .

2) Relations between α and i

Throughout these notes $i \in E(M,\mathbb{R}^n)$ is a fixed (smooth) embedding and $\alpha \in A^1(M,\mathbb{R}^n)$ a fixed (smooth) $\mathbb{R}^n$-valued one form $di \in A^1(M,\mathbb{R}^n)$ is locally given by the Frêchet derivative of i. Clearly $m(i)(X,Y)=<di\, X, di\, Y>$ for all $X,Y \in \Gamma TM$.

Our first observation is as follows:

<u>Proposition 1</u> Given $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$ then there is a map $h(i) \in C^\infty(M,\mathbb{R}^n)$, determined up to a constant such that

(1) $$\alpha = dh(i) + \beta(i)$$

$h(i)$ is called the integrable part of α. The decomposition (1) of α is maximal in the sense that the integrable part of $\beta(i)$ is zero.

<u>Proof:</u> Let $e_1,\ldots,e_n$ be an orthonormal basis of $\mathbb{R}^n$. Then

$$\alpha(X) = \sum_{s=1}^{n} \alpha^s(X) e_s \qquad \forall\, X \in \Gamma TM$$

for a uniquely determined family $\alpha^1,\ldots,\alpha^n \in A^1(M,\mathbb{R})$ of smooth $\mathbb{R}$-valued one forms of M. Clearly

$$\alpha^s(X) = < \alpha(X), e_s > .$$

For each $s = 1,\ldots,n$ the one form α^s can be represented as

$$\alpha^s(X) = m(i)(Y_s, X) \quad \text{for all } X \in \Gamma TM$$

for a well defined $Y_s \in \Gamma TM$. This vector field splits according to Hodge's decomposition uniquely into

$$Y_s = \mathrm{grad}_i \tau_s + Y_s^o$$

where $\mathrm{grad}_i \tau_s$ means the gradient of $\tau_s \in C^\infty(M,\mathbb{R})$ with respect to the metric $m(i)$ and $\mathrm{div}_i Y^o = 0$. By div_i we denote the divergence operator associated with $m(i)$. Thus

$$\alpha^s(X) = d\tau_s(X) + m(i)(Y_s^o,X) \quad .$$

Now we define

$$h(i) := \sum_{s=1}^{n} \tau_s \cdot e_s$$

and

$$\beta(i)(X) := \sum_{s=1}^{n} m(i)(Y_s^o,X)\cdot e_s \qquad \forall\, X \in \Gamma TM \quad .$$

Then

$$\alpha = dh(i) + \beta(i) \quad .$$

Let us verify that this decomposition does not depend on the basis chosen. To this end let $\bar{e}_1,\ldots,\bar{e}_n \in \mathbb{R}^n$ be any other orthonormal basis of $\mathbb{R}^n$ and define $\bar{\alpha}^s, \bar{\tau}_s, \bar{Y}_s^o$, $\bar{h}$ and $\bar{\beta}$ analoguously as above. Omiting the variable i then for any $\bar{s} = 1,\ldots,n$

$$\bar{\alpha}^{\bar{s}}(X) = \langle \alpha(X), \bar{e}_{\bar{s}} \rangle = \langle dh(X), \bar{e}_s\rangle + \langle \beta(X), \bar{e}_s\rangle =$$
$$= \langle \sum_1^{n} d\tau_s(X)\cdot e_s, \bar{e}_{\bar{s}} \rangle + \langle \sum_i^{n} m(i)(Y_s^o,X)\cdot e_s, \bar{e}_{\bar{s}} \rangle$$
$$= m(i)(\mathrm{grad}_i \sum_1^{n})(\tau_s \cdot \langle e_s \bar{e}_{\bar{s}}\rangle),X) + m(i)(\sum_1^{n} Y_s^o \cdot \langle e_s, \bar{e}_{\bar{s}}\rangle, X)$$
$$= \langle d\bar{h}(X), \bar{e}_{\bar{s}} \rangle + \langle \beta(X), \bar{e}_{\bar{s}} \rangle +$$
$$= m(i)(\mathrm{grad}_i\, \bar{\tau}_{\bar{s}}, X) + m(i)(Y_{\bar{s}}^o, X) \quad .$$

Since $\mathrm{grad}_i \sum_1^{n} (\tau_s \langle e_s, e_{\bar{s}}\rangle)$ is a gradient with respect to $m(i)$ and since

$$\mathrm{div}_i(\Sigma\, Y_s^o \langle e_s, \bar{e}_{\bar{s}}\rangle) = o$$

we conclude due to the uniqueness of Hodge's decomposition

$$\mathrm{grad}_i \sum_{s=1}^{n} (\tau_s \cdot \langle e_s, e_{\bar{s}}\rangle) = \mathrm{grad}_i\, \bar{\tau}_s$$

and

$$\sum_{s=1}^{n} Y_s^o \langle e_s, \bar{e}_{\bar{s}} \rangle = Y_{\bar{s}}^o \quad .$$

Thus for any $X \in \Gamma TM$ we have

$$dh(X) = \sum_{\bar{s}=1}^{n} \langle dh(X), \bar{e}_s\rangle\, \bar{e}_s = \sum_{\bar{s}=1}^{n} \langle d\bar{h}(X), \bar{e}_s\rangle \cdot \bar{e}_s = d\bar{h}(X)$$

and

$$\beta(X) = \sum_{\bar{s}=1}^{n} \langle \beta X), \bar{e}_s \rangle \cdot \bar{e}_s = \Sigma \langle \bar{\beta}(X), \bar{e}_{\bar{s}} \rangle \cdot \bar{e}_s = \bar{\beta}(X) \quad .$$

Let us investigate the decomposition (1) in proposition 1 somewhat closer. We have

$$\alpha = dh(i) + \beta(i)$$

for some $h \in C^\infty(M,\mathbb{R}^n)$. Obviously

$$h(i) = di\, X_h + h^\perp(i)$$

for some well-defined $X_h \in \Gamma TM$. By $h^\perp(i)$ we denote the pointwise formed component of h normal to $i(M)$. The vector field X_h again decomposes according to Hodge's decomposition into

$$X_h = X_h^o + \text{grad}_i\, \psi_h \quad \text{with} \quad \text{div}\, X_h^o = o \quad \text{and} \quad \psi_h \in C^\infty(M,\mathbb{R}) .$$

Hence
$$dh(i)X = di\, \nabla(i)_X X_h^o + di(\nabla(i)_X\, \text{grad}_i\, \psi_h + W(i)_h\, X)$$
$$+ S(i)(X_h,X) .$$

Here $diW(i)_h\, X = dh^\perp(i)(X)^T$, where the upper indices T and $\perp$ denote the pointwise formed component tangential respectively normal to $i(M)$. We remind the reader that $W_h(i)$ is a smooth strong bundle endomorphism of TM selfadjoint with respect to $m(i)$. Let now $h^1 = h + z$ for some $z \in \mathbb{R}^n$. Then if we regard z as a constant map in $C^\infty(M,\mathbb{R})$ we again have

$$z = di\, X_z + z^\perp .$$

However the vector field on $\mathbb{R}^n$ assigning to any $z' \in \mathbb{R}^n$ the vector $z \in \mathbb{R}^n$ is a gradient of some map $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ say . Hence if we form $\varphi \circ i$ then

$$X_z = \text{grad}_i(\varphi \circ i) .$$

Therefore
$$h^1(i)^T = di\, X_h^o + \text{grad}_i(\psi_h + \varphi \circ i)$$
$$= di\, X_{h^1}^o + \text{grad}_i\, \psi_{h^1} .$$

Again due to the uniqueness of Hodge's decomposition of X_{h^1} we conclude:

<u>Proposition 2</u> Given $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$, then if the splitting $\alpha = dh(i) + \beta(i)$ is maximal

for some $h(i) \in C^\infty(M,\mathbb{R}^n)$ determined up to a constant and if for some $h^1(i) \in C^\infty(M,\mathbb{R}^n)$ with $h^1(i) = h(i)+z$ and $z \in \mathbb{R}^n$

$$\alpha = dh^1(i) + \beta(i)$$

then

$$X_h^o = X_{h^1}^o .$$

Here X_h^o and $X_{h^1}^o$ denotes the divergence free part of X_h and X_{h^1} respectively.

Next we will study $\alpha \in A^1(M,\mathbb{R}^n)$ in relation to a fixed $i \in E(M,\mathbb{R}^n)$ from a quite different point of view.

For any pair X,Y we set

$$T(\alpha,i)(X,Y) := \langle \alpha(X), di\, Y \rangle$$

Hence $T(\alpha,i)$ is a smooth two tensor on M, splitting uniquely into a symmetric and an antisymmetric part $T(\alpha,i)^s$ and $T(\alpha,i)^a$ respectively. If $P : TM \longrightarrow TM$ denotes the unique smooth strong bundle endomorphism for which

$$T(\alpha,i)(X,Y) = m(i)(PX,Y)$$

then

$$T(\alpha,i)^s(X,Y) = m(i)\left(\tfrac{1}{2}(P + \tilde{P})X,Y\right)$$

and

$$T(\alpha,i)^a(X,Y) = m(i)\left(\tfrac{1}{2}(P - \tilde{P})X,Y\right) .$$

Here $\tilde{P}$ denotes the fibrewise formed adjoint of P with respect to $m(i)$. Let us set

$$B_\alpha(i) = \tfrac{1}{2}(P + \tilde{P}) \quad \text{and} \quad C_\alpha(i) = \tfrac{1}{2}(P - \tilde{P}) .$$

Therefore

$$\alpha(X) = \alpha^1(X) + di\, C_\alpha X + di\, B_\alpha X$$

holds for any $X \in \Gamma TM$. Clearly $\alpha^1(X)(p)$ is a vector in the normal space of $Ti\, T_pM$ for each $p \in M$. Hence there is a unique smooth

$$c_\alpha \in C^\infty(M, so(n))$$

where $so(n)$ denotes the Lie algebra of $SO(n)$ such that

$$\alpha^1(X) = c_\alpha \cdot di\, X \qquad \forall\, X \in \Gamma TM .$$

Thus we may state that a second decomposition of α relative to i :

<u>Proposition 3</u> Given $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$ there are uniquely determined smooth, strong bundle endomorphisms

$$C_\alpha(i) : TM \longrightarrow TM$$

and

$$B_\alpha(i) : TM \longrightarrow TM$$

which are with respect to $m(i)$ skew-respectively selfadjoint and there is a uniquely determined $c_\alpha(i) \in C^\infty(M,so(n))$ such that the following relation holds for all $X \in \Gamma TM$:

(2) $$\alpha(X) = c_\alpha(i)\cdot di\, X + di\cdot C_\alpha(i)X + di\cdot B_\alpha(i)X .$$

<u>Remark:</u> Given $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$ then the exterior differential $\partial T(\alpha,i)^a$ of $T(\alpha,i)^a$ satisfies

(3) $$\partial T^a(\alpha,i) = o \quad \text{iff} \quad \partial\alpha = o .$$

The reason is that the one-form $< i,\alpha > \in A^1(M,\mathbb{R})$ assigning to any $X \in \Gamma TM$ the function $<i,\alpha(X)>$ satisfies

$$\partial < i,\alpha > = T(i,\alpha)^a \quad \text{iff} \quad \partial\alpha = o .$$

Now we will link the two descriptions of α relative to i as expressed by the two propositions (1) and (2). To this end let $\alpha \in A^1(M,\mathbb{R}^n)$ and

$i \in E(M,\mathbb{R}^n)$ be given. Let

$$\alpha = dh(i) + \beta(i)$$

be the decomposition described in proposition (1). We split $h(i)$ into

$$h(i) = di\, X_h + h^{\perp}(i) \quad .$$

Hence for any $Y \in \Gamma TM$

(4) $$dh(i)Y = di\, \nabla(i)_Y X_h + W_h(i)Y + $$
$$+ S(i)(Y,X_h) + (d(h^{\perp})(Y))^{\perp} + \beta(Y) \quad .$$

Forming $T(dh,i)$, decomposing it into $T(dh,i)^s$ and $T(dh,i)^a$ and using (4) yields immediately

$$T(dh,i)^s(X,Y) = m(i)(X,\nabla(i)_Y X_h) + m(i)(X,W_h(i)Y) \quad .$$

Therefore

$$T(dh,i)^a(X,Y) = \tfrac{1}{2}(m(i)(X,\nabla(i)_Y X_h) - m(i)(Y,\nabla(i)_X X_h)$$

and
$$T(dh,i)^s(X,Y) = \tfrac{1}{2}(m(i)(X,\nabla(i)_Y X_h) + m(i)(Y,\nabla(i)_Y X_h)$$
$$+ m(i)(X,W_h(i)Y) = \tfrac{1}{2}L_{X_h}(m(i))(X,Y) + m(i)(X,W_h(i)Y)$$

Here $L_{X_h}(m(i))$ is the Lie derivative of $m(i)$. Writing for any $Z \in \Gamma TM$

$$L_{Z_h}(m(i))(X,Y) = m(i)(\mathbb{L}_{Z_h} X,Y)$$

where
$$\mathbb{L}_Z : TM \longrightarrow TM$$

is a strong smooth bundle endomorphism given by the theorem of Fischer-Riesz, then $c_\alpha(i)$, $C_\alpha(i)$ and $B_\alpha(i)$ relate to h as follows

(5) $$c_\alpha(i)di\, Y = (d(dh^{\perp}(i))Y)^{\perp} + S(i)(Y,X_h) + C_\alpha(i)\cdot di\, Y$$

(6) $$C_\alpha(i)Y = \tfrac{1}{2}\,(\nabla(i)X_h - \check{\nabla}(i)X_h)Y + C_\beta(i)Y$$

and
$$B_\alpha(i)Y = \tfrac{1}{2}\,(\nabla(i)X_h + \check{\nabla}(i)X_h)Y + W(i)_h Y + B_\beta(i)Y =$$
$$= (\tfrac{1}{2}\,\mathbb{L}_{X_h}(i) + W(i)_h + B_\beta(i))Y \quad .$$

Here $\tilde{\nabla}(i)\, X_h$ means the fibrewise formed adjoint with respect to $m(i)$, which applied to $v_p \in T_pM$ is written as $\tilde{\nabla}(i)X_h(v_p)$ for any $p \in M$. If we split furthermore X_h into

$$X_h = X_h^o + \mathrm{grad}_i\psi \quad \text{with} \quad \mathrm{div}_i X_h^o = o$$

(according to Hodge's decomposition) and taking

$$0 = m(i)((\nabla(i)\mathrm{grad}_i\psi - \tilde{\nabla}(i)\ \mathrm{grad}_i\psi)\, X,Y) = 0$$

into account yields finally the desired relations

Proposition 4 Let $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$. Given any $h(i) \in (M,\mathbb{R}^n)$ with

$$h(i) = di\ X_h(i) + h^\perp = X_h(i)^o + \mathrm{grad}_i\psi(i) + h^\perp$$

as split according to the Hodge decomposition of X_h and

$$\alpha = dh(i) + \beta(i)$$

then the coefficients in

$$\alpha = c_\alpha(i)\cdot di + di\ C_\alpha(i) + di\ B_\alpha(i)$$

are determined by

(8) $$c_\alpha(i)\cdot di = d(dh^\perp(i))^\perp + S(i)(X_h(i),\ldots) + c_\beta\cdot di$$

(9) $$C_\alpha(i) = \tfrac{1}{2}(\nabla(i)X_h(i) - \tilde{\nabla}(i)X_h(i)) + C_\beta(i)$$
$$= \tfrac{1}{2}\ (\nabla(i)X_h(i)^o - \tilde{\nabla}(i)X_h(i)^o) + C_\beta(i)$$

and

(10) $$B_\alpha(i) = \tfrac{1}{2}\ \mathbb{L}_{X_h(i)^o + \mathrm{grad}_i\psi(i)} + W_h(i) + B_\beta\ .$$

Hence

(11) $$\mathrm{tr}\ B_\alpha(i) = \mathrm{div}\ X_h(i) + \mathrm{tr}\ W_h(i) + \mathrm{tr}\ B_\beta =$$
$$= -\ \nabla(i)\psi(i) + \mathrm{tr}\ W_h(i) + \mathrm{tr}\ B_\beta$$

with $\nabla(i)$ is the Laplace Beltrami operator of $m(i)$.

The rest of this section is devoted to the covariant divergence of $B_h(i)$ and $C_h(i)$. The covariant divergence $\mathrm{div}_i A$ of any smooth strong bundle endomorphism

$$A : TM \longrightarrow TM$$

is defined as follows: Let $e_1,\ldots,e_m$ be any moving orthonormal frame of TM. Then

$$\mathrm{div}_i\, A = \sum_{r=1}^{r=m} \nabla(i)_{e_r}(A)\, e_r \quad . \tag{12}$$

First we compute $\mathrm{div}_i\, \nabla(i)X_h$. For any $Y \in \Gamma TM$ the equation

$$m(i)(\nabla(i)_{e_r}(\nabla(i)X_h)e_r,Y) = m(i)(\nabla(i)_{e_r}(\nabla(i)_{e_r}X_h)-\nabla(i)_{\nabla(i)_{e_r}e_r}X_h,Y)$$

implies

$$\mathrm{div}_i\, \nabla(i)X_h = -\,\Delta(i)X_h \tag{13}$$

$\Delta(i)$ being the Laplace Beltrami operator of $m(i)$. To find $\mathrm{div}_i\, \tilde{\nabla}(i)X_h$ consider for any $Y \in \Gamma TM$ the equations

$$\begin{aligned} m(i)(\nabla(i)_{e_r}(\tilde{\nabla}(i)X_h)e_r,Y) &= e_r(m(i)(\tilde{\nabla}(i)X_h(e_r),Y)) - m(i)(\tilde{\nabla}(i)X_h(\nabla(i)_{e_r}e_r),Y) \\ &\quad - m(i)(\tilde{\nabla}(i)X_h(e_r),\, \nabla(i)_{e_r}Y) \\ &= m(i)(e_r,\nabla(i)_{e_r}\nabla(i)_Y X_h) - m(i)(e_r,\nabla(i)_{\nabla(i)_{e_r}Y}X_h) \\ &= m(i)(e_r,\nabla(i)_{e_r}(\nabla(i)X_h)Y) \end{aligned}$$

and

$$\begin{aligned} m(i)(\nabla(i)_Y(\tilde{\nabla}(i)X_h)e_r,e_r) &= m(i)(e_r,\nabla(i)_Y\nabla(i)_{e_r}X_h) - m(i)(e_r,\nabla(i)_{\nabla(i)_Y e_r}X_h) \\ &= m(i)(e_r,\nabla(i)_Y(\nabla(i)X_h)e_r) \ . \end{aligned}$$

Thus we find

$$\sum_{r=1}^{m} (m(i)(\nabla(i)_{e_r}(\tilde{\nabla}(i)X_h)e_r,Y) - m(i)(\nabla(i)_Y(\tilde{\nabla}(i)X_h)\,e_r,e_r)) =$$

$$= \mathrm{Ric}(m(i))(Y,X_h)$$

and consequently

$$m(i)(\mathrm{div}_i\, \tilde{\nabla}(i)X_h, Y) = \mathrm{tr}\, \nabla(i)_Y(\nabla(i)X_h) + \mathrm{Ric}(m(i))(Y, X_h) \;.$$

Here $\mathrm{Ric}(m(i))$ denotes the Ricci tensor of $m(i)$. The last equation yields

(14) $$\mathrm{div}_i \tilde{\nabla}(i)X_h = \mathrm{grad}_i\, \mathrm{div}_i\, X_h + R(i)X_h \;.$$

Here $\mathrm{Ric}(i)X_h$ is defined via:

$$m(i)(R(i)X_h, Y) = \mathrm{Ric}(m(i))(X_h, Y) \qquad \forall\, Y \in \Gamma TM \;.$$

Now we immediately conclude

(15) $$\mathrm{div}\, \mathbb{L}_{X_h}(i) = -\,\Delta(i)X_h + R(i)X_h + \mathrm{grad}\, \mathrm{div}\, X_h$$

(16) $$2\, \mathrm{div}\, C_h(i) = -\,\Delta(i)X_h - R(i)X_h - \mathrm{grad}\, \mathrm{div}\, X_h$$

showing

(17) $$\mathrm{div}_i(\tfrac{1}{2}\, \mathbb{L}_{X_h}(i) + C_h(i)) = -\,\Delta(i)\, X_h$$

and

(18) $$\mathrm{div}(\tfrac{1}{2}\, \mathbb{L}_{X_h}(i) - C_h(i) = \mathrm{Ric}(i)\, X_h + \mathrm{grad}_i\, \mathrm{div}_i\, X_h \;.$$

These equations will be of interest later.

Let us restrict our attention to the case of

$$1 + \dim M = n \;.$$

Then since M is oriented we have the oriented unite normal vector field $N(i)$ along i. Hence $h \in C^\infty(M, \mathbb{R}^n)$ splits into

$$h(i) = di\, X_h + \tau(i)\cdot N(i)$$

for some $\tau(i) \in C^\infty(M, \mathbb{R})$. Thus $W_h(i) = W(i)$ if $\tau = 1$. Denoting $\mathrm{tr}\, W(i)$ by $H(i)$ we immediately find

$$m(i)(\mathrm{div}_i(\tau(i)\cdot W(i)),Y) = \sum_{r=1}^{m} m(i)(\nabla(i)_{e_r}(\tau(i)W(i))e_r,Y) =$$

$$= \sum_{r=1}^{m} m(i)(\mathrm{grad}_i\ \tau(i),\ W(i)Y) + m(i)(\tau(i)\mathrm{div}_i\ W(i),Y)$$

and hence

$$(19) \qquad \mathrm{div}_i(\tau(i)W(i)) = W(i)\mathrm{grad}_i\ \tau(i) + \tau(i)\mathrm{div}_i W(i)\ .$$

Now by Codazzi's equation (cf.[Kl]) yields

$$(20) \qquad \sum_{r=1}^{m} m(i)(\nabla(i)_{e_r}(W(i))e_r,Y) = \sum_{r=1}^{m} m(i)(\nabla(i)_Y(W(i))e_r,e_r) =$$

$$= m(i)(\mathrm{grad}\ H(i),Y)\ .$$

We therefore have

$$(21) \qquad \mathrm{div}_i\tau(i)W(i) = W(i)\mathrm{grad}_i\ \tau(i) + \tau(i)\mathrm{grad}_i\ H(i)\ .$$

In turn equations (19),(20),(9),(10),(15),(16),(17) and (18) yield

$$(22) \qquad \mathrm{div}_i(B_h(i) + C_h(i)) = -\Delta(i)X_h + W(i)\mathrm{grad}_i\ \tau(i)+\tau(i)\mathrm{grad}_i H(i)$$

and

$$(23) \qquad \mathrm{div}(B_h(i) - C_h(i)) = R(i)\ X_h + \mathrm{grad}_i\ \mathrm{div}_i\ X_h +$$
$$+\ W(i)\mathrm{grad}_i\ \tau(i) + \tau(i)\mathrm{grad}_i\ H(i)\ .$$

We close this section by showing the following result:

<u>Lemma 5</u> Let $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$. If α has no integrable part then $\alpha = \beta$ and hence

$$(24) \qquad \mathrm{div}_i(C_\beta(i) + B_\beta(i)) = o\ .$$

<u>Proof:</u> We have for any $X \in \Gamma TM$

$$\alpha(X) = \beta(X) = \sum_{s=1}^{m} m(i)(Y_s^{\,o},X)\bar{e}_s$$

and any orthonormal frame $\bar{e}_1,\dots,\bar{e}_n$ in $\mathbb{R}^n$.

Hence

$$< \alpha(X), di\, Y > = m(i)((C_\beta(i) + B_\beta(i))X,Y) = \sum_{s=1}^{m} m(i)(Y_s^{\,o},X) < \bar{e}_s, di\, Y > .$$

Thus if $e_1,\dots,e_m$ is a moving orthonormal frame in TM , then

$$m(i)(\mathrm{div}_i(C_\beta(i)+B_\beta(i)),Y) = \sum_{r=1}^{m} \sum_{s=1}^{m} m(i)(\nabla(i)_{e_r} Y_s^{\,o}, e_r) < \bar{e}_s, di\, Y >$$

Interchanging the summation yields (24).

Therefore we have due to (24) and (22)

<u>Corollary 6</u> Let $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$ and let $\alpha = dh+\beta$ as in (1). If $h(i) = di\, X_1 + \tau \cdot N(i)$ then

$$\mathrm{div}_i(B_\alpha(i) + C_\alpha(i)) = \mathrm{div}(B_h(i) + C_h(i)) =$$
$$= - \Delta(i)X_h + W(i)\mathrm{grad}_i\, \tau + \tau \cdot \mathrm{grad}_i H(i) .$$

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