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Abstract

By an $\mathbb{R}^n$-invariant constitutive law F we mean a smooth $\mathbb{R}^n$-invariant one form on the Fréchet manifold $E(M,\mathbb{R}^n)$ of all Euclidean smooth embeddings of a compact manifold M. Associated with it are a natural integrable $\mathbb{R}^n$-valued one form and a natural two tensor, both embedding dependent, provided F is induced by a one form $\tilde{F}$ on $E(M,\mathbb{R}^n)/\mathbb{R}^n$ and $\tilde{F}$ admits an integral representation. This two tensor plays the role of the stress tensor in elasticity theory.

Introduction

Let M be a compact smooth and oriented manifold and $E(M,\mathbb{R}^n)$ the Fréchet manifold of all smooth embeddings of M into $\mathbb{R}^n$, $E(M,\mathbb{R}^n)$ carrying the C^∞-topology. For the sake of simplicity we choose $\dim M = n-1$.

First we consider smooth $\mathbb{R}$-valued one forms on $E(M,\mathbb{R}^n)$ and integral representations of a certain kind. These representations are unique if they exist. (The smoothness is ment in the sense of [Gu] or [Mi]).

If a one form F on $E(M,\mathbb{R}^n)$ is of the form $F = d^*\tilde{F}$ where $\tilde{F}$ is a one form on $E(M,\mathbb{R}^n)/\mathbb{R}^n$ then F is $\mathbb{R}^n$-invariant. The main purpose of these notes is to show that in case $\tilde{F}$ admits an integral representation then $\tilde{F}(dj,\ldots)$ yields in a natural way a differential dh and a two tensor T both depending on dj. Here $d: E(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}^n)/\mathbb{R}^n$ is the usual differential, which locally represented is nothing else but the Fréchet differential of j. It is dh which determines $\tilde{F}$ fully and not T.

The integral representation of $\tilde{F}$ by $\mathbb{R}^n$-valued one forms on M are unique provided the integrable part of the $\mathbb{R}^n$-valued form is used only.

In case of elasticity theory $E(M,\mathbb{R}^n)$ is the configuration space of the body M (cf. [Ma,Hu]). F corresponds to a constitutive law (which describes the virtual work) and if it admits an integral representation, then it is represented by the force density. If F is of the form d^*F and if $\tilde{F}$ admits an integral representation then T plays the role of the (not necessarily symmetric) stress tensor. Clearly $\tilde{F}$ and hence T itself may depend on further parameters.

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Constitutive laws

Let M be a compact smooth and oriented manifold of dimension $n-1$ and $\langle,\rangle$ be a fixed scalar product on the oriented $\mathbb{R}^n$. By $E(M,\mathbb{R}^n)$ we mean the set of all smooth embeddings of M into $\mathbb{R}^n$. This set is open in $C^\infty(M,\mathbb{R}^n)$, the collection of all smooth $\mathbb{R}^n$-valued maps of M endowed with the C^∞-topology (cf. [Hi]). Under the pointwise defined operations $C^\infty(M,\mathbb{R}^n)$ is a Fréchet space and hence $E(M,\mathbb{R}^n)$ is a Fréchet manifold. Obviously

$TE(M,\mathbb{R}^n) = E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n)$.

Given moreover $j \in E(M,\mathbb{R}^n)$ the Riemannian metric $m(j)$ is defined by

$$m(j)(X,Y) = \langle djX, djY\rangle$$

$\forall$ X,Y in ΓTM, the $C^\infty(M,\mathbb{R})$-module of all smooth vector fields on M. By dj we mean the second factor in $Tj = (j,dj)$ locally given by the Fréchet derivative of j. Let moreover $\nabla(j)$ and $\mu(j)$ be the Levi-Cività connection and the Riemannian volume form of m(j) respectively. Finally let us denote by $A^1(N,\mathbb{R}^m)$ the collection of all $\mathbb{R}^m$-valued one forms of N whatever the manifold N and the natural number m meight be. This space is endowed with the C^∞-topology.

By a constitutive law we mean a smooth one form F on $E(M,\mathbb{R}^n)$, that is a smooth map

$$F\colon E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$$

for which

$$F(j)\colon C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$$

assignig to each $k \in C^\infty(M,\mathbb{R}^n)$ the value $F(j,k)$ is linear for any $j \in E(M,\mathbb{R}^n)$. (Obviously F may depend on further parameters).

We call F $\mathbb{R}^n$-invariant provided that

$$F(j+z,k) = F(j,k)$$

for any $z \in \mathbb{R}^n$ regarded as a function in $C^\infty(M,\mathbb{R}^n)$ assuming z as its only value. Let us assume that F factors over $E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \times C^\infty(M,\mathbb{R}^n)_{\mathbb{R}^n}$. Both factors of this Cartesian product are Fréchet manifolds when endowed with the C^∞-topology. Evidently the differential

$$d\colon C^\infty(M,\mathbb{R}^n) \longrightarrow \{dh \mid h \in C^\infty(M,\mathbb{R}^n)$$

factors over $C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ and yields a bijection onto its range, which hence is a Fréchet manifold diffeomorphic to $C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$. Thus F is given by some

$$\tilde{F} \in A^1(E(M,\mathbb{R}^n)/_{\mathbb{R}^n},\mathbb{R})$$

as $F = d^*\tilde{F}$ and is thus $\mathbb{R}^n$-invariant.

Integral representation and stress tensor

Let $\gamma \in A^1(M,\mathbb{R}^n)$ and $j \in E(M,\mathbb{R}^n)$. With these dates we may form the two tensor $T(\gamma,j)$ defined by

$$T(\gamma,j)(X,Y) := \langle\gamma(X),djY\rangle$$

for all $X,Y \in \Gamma TM$.

There is a unique smooth strong bundle endomorphism $P(j)$ of TM such that

$$T(\gamma,j)(X,Y) = m(j)(P(j)X,Y).$$

Decomposing $P(j)$ into its skew, and selfadjoint part with respect to $m(j)$ denoted $C_\gamma(j)$ and $B_\gamma(j)$, respectively yields

$$T(\gamma,j)(X,Y) = \langle dj(C_\gamma(j) + B_\gamma(j)(X),djY\rangle.$$

Hence γ is uniquely represented as

$$\gamma(X) = c_\gamma(j)diX + dj\ C_\gamma X + dj\ B_\gamma X$$

where $c_\gamma(j) \in C^\infty(M,so(n))$ ($so(n)$ being the Lie algebra of $SO(n)$) maps for any $p \in M$ diT_pM into $V_p(j)$, the normal space of diT_pM and vice versa. We now define for any choice of $\beta,\gamma \in A^1(M,\mathbb{R}^n)$ and a given $j \in E(M,\mathbb{R}^n)$ their dot product $\beta\cdot\gamma$ by setting

$$\beta\cdot\gamma := -\mathrm{tr}\ c_\beta(j)\circ c_\gamma(j) - \mathrm{tr}\ C_\beta(j)\circ C_\gamma(j) + \mathrm{tr}\ B_\beta(j)B_\gamma(j).$$

We say that $\tilde{F} \in A^1(E(M,\mathbb{R}^n)/_{\mathbb{R}^n},\mathbb{R})$ admits an integral representation by $\alpha \in C^\infty(E(M,\mathbb{R}^n),A^1(M,\mathbb{R}^n))$ if

$$\tilde{F}(j,k) = \int \alpha(dj)\cdot dk\ \mu(j)\ \forall\ j \in E(M,\mathbb{R}^n) \text{ and } \forall\ k \in C^\infty(M,\mathbb{R}^n).$$

$\alpha(dj)$ is called the stress form of $F(j)$ and $T(\alpha(dj),j)$ the stress tensor associated with $F(j) = d^*\tilde{F}$.

The stress form α and hence the associated stress tensor reflect obviously $\mathbb{R}^n$-invariant, and hence in particular, internal physical properties of the moving body.

First we prove the following:

Theorem 1: If $\tilde{F}$ admits an integral representation by $\alpha \in C^\infty(E(M,\mathbb{R}^n),A^1(M,\mathbb{R}^n))$ then $F := d^*\tilde{F}$ admits a unique integral representation as

$$F(j,k) = \int \langle\varphi(j),k\rangle\mu(j)\ \forall\ j \in E(M,\mathbb{R}^n) \text{ and } \forall\ k \in C^\infty(M,\mathbb{R}^n)$$

with "force density" $\varphi(j)$ determined by α via

$$\varphi(j) = -\mathrm{div}(B_\alpha(j) + C_\alpha(j)) - 2\cdot W(j)U_\alpha(j) + + (\mathrm{tr}\ B_\alpha(j)\circ W(j))\cdot N(j)$$

where $U_\alpha(j)$ is defined by $dj\ U_\alpha(j)X = c_\alpha(j)djX$ for any $X \in \Gamma TM$ and where $W(j)$ is the Weingarten map of j.

Proof: From above we know that dk writes uniquely as

$$dk = c_k\cdot di + di\cdot C_k + di\cdot B_k.$$

Since $k = djX_k + \Psi_k\cdot N(j)$ for some $X_k \in \Gamma TM$, some $\Psi_k \in C^\infty(M,\mathbb{R}^n)$ and the oriented unit normal vector field $N(j)$ the coefficients B_k, C_k and c_k are of the form

$$B_k = ½\cdot(\nabla(j)X_k + \tilde{\nabla}(j)X_k) + \Psi_k\cdot W(j)$$

$\tilde{\nabla}(j)X_k$ being the pointwise formed adjoint of $\nabla(j)X_k$ with respect to $m(j)$, moreover

$$C_k = ½\cdot(\nabla(j)X_k - \tilde{\nabla}(j)X_k)$$

and

$$c_k\cdot dj\ Y = S(j)(X_k,Y) + d\Psi_k(X)\cdot N(j)$$

for any $Y \in \Gamma TM$. $S(j)$ is the $\mathbb{R}^n$-valued second fundamental tensor of j, that is the pointwise formed component of $d(djX_k)(Y)$ normal to $j(M)$. Let us write c, C and B instead of $c_\alpha(j)$, $C_\alpha(j)$ and $B_\alpha(j)$ respectively. Thus we have for an orthonormal moving frame $e_1,\ldots,e_{n-1}$ and $\Psi_k = 1$

$$\mathrm{tr}\ B\circ B_{dk}(j) = ½ \sum_{s=1}^{n-1} m(j)(B\circ(\nabla(j)X_k + \tilde{\nabla}(j)X_k)e_s,e_s)$$

$$= \sum_s m(j)(B(\nabla(j)_{e_s}X_k),e_s) - \sum_s \langle N(j),S(j)(e_s,Be_s)\rangle$$

$$= \mathrm{div}_j(B\ X_k) - m(j)(\mathrm{div}_j\ B,X_k) + \langle(\mathrm{tr}\ B\circ W(j))\cdot N(j),N(j)\rangle$$

with

$$\mathrm{div}_j\ A = \sum_{s=1}^{n-1} \nabla(j)_{e_s}(A)e_s$$

for any strong smooth bundle endomorphism of TM. A similar expression holds for $\mathrm{tr}\ C\circ C_{dk}$ which to calculate we omit to the reader. Next we find

$$\mathrm{tr}\ c\circ c_{dk} = \sum_{s=1}^{n-1} \langle c\circ c_{dk}\ dj\ e_s,\ dj\ e_s\rangle + \langle c\circ c_{dk}N(j),N(j)\rangle =$$

$$= -\sum_s m(j)(W(j)X_k,e_s)\ \langle c\ N(j),dj\ e_s\rangle - m(j)(W(j)U_\alpha(j),X_k) =$$

$$= -\sum_s m(j)(W(j)X_k,e_s)m(j)(U_\alpha(j),e_s) - m(j)(W(j)U_\alpha(j),X_k) =$$

$$= -2\ m(j)(W(j)U_\alpha(j),X_k).$$

Using Gausses theorem one reads off immediately the expression for $\varphi(j)$.

The influence of the stress form and the stress tensor on the constitutive law

Let $\gamma \in A^1(M,\mathbb{R}^n)$ and $j \in E(M,\mathbb{R}^n)$. We know (cf. [Bi]) that γ uniquely splits into

$$\gamma = dh + \beta$$

where dh with $h \in C^\infty(M,\mathbb{R}^n)$ is an integrable part of γ and β admits zero as integrable parts only. dh is therefore called the integrable and β the non-integrable part of γ. This decomposition of γ is of the form

$$dhX = \sum_{s=1}^{n} m(j)(\mathrm{grad}_j \tau_s, X)\cdot \bar{e}_s \quad \forall\ X \in \Gamma TM$$

and

$$\beta(X) = \sum_{s=1}^{n} m(j)(Y_s^o, X)\cdot \bar{e}_s \quad \forall\ X \in \Gamma TM$$

where $\bar{e}_1,\dots,\bar{e}_n$ is any basis in $\mathbb{R}^n$, $\tau_s \in C^\infty(M,\mathbb{R})$ and $Y_s^o \in \Gamma TM$ with $\mathrm{div}_j Y_s^o = 0$ for all $s = 1,\dots,n$. Here $\mathrm{div}_j Y_s^o = \mathrm{tr}\ \nabla(j)Y_s^o$.

As a consequence of this decomposition we have the following:

Theorem 2 Given $\gamma \in A^1(M,\mathbb{R}^n)$ and $j \in E(M,\mathbb{R}^n)$ with dh as the integrable part of γ then

$$\int \gamma\cdot dk\ \mu(j) = \int dh\cdot dk\ \mu(j),$$

saying that the non-integrable part β of $\gamma = dh+\beta$ is orthogonal to $C^\infty(M,\mathbb{R}^n)/\mathbb{R}^n$, regarded as a subspace of $A^1(M,\mathbb{R}^n)$.

Proof: Simplifying the notion from above let

$$dhX = \sum_s m(j)(V_s,X)e_s \quad \forall\ X \in \Gamma TM$$

where $V_s \equiv \mathrm{grad}_j \tau_s$ for all s. Then

$$(B_h + C_h)X = \sum_s m(j)(V_s,X)X_s$$

and

$$(B_\beta + C_\beta)Y = \sum_s m(j)(Y_s^o,Y)X_s.$$

Here $X_s \in \Gamma TM$ for any $s = 1,\dots,n$ are such that

$m(j)(X_s,Z) = \langle \bar{e}_s, djZ\rangle$ for all $Z \in \Gamma TM$.

Hence

$$(B_h + C_h)\circ(B_\beta + C_\beta)X = \sum_{s'}\sum_{s} m(j)(Y_s^o,X)m(j)(V_{s'},X_s)X_s.$$

Therefore if $e_1,\ldots,e_{n-1}$ is an orthonormal moving frame on M

$$\operatorname{tr}(B_h + C_h)\circ(B_\beta + C_\beta) = \operatorname{tr}(B_h\circ B_\beta + C_h\circ C_\beta)$$

$$= \sum_r \sum_{s'} \sum_s m(j)(Y_s^o,e_r)m(j)(V_{s'},X_s)m(j)(X_{s'},e_r)$$

$$= \sum m(j)(Y_s^o,V_s).$$

Next we form

$$c_h\, diX = \sum_s m(j)(V_s,X)\cdot\langle e_s,N(j)\rangle\cdot N(j)$$

$$c_h\, N(j) = \sum_s \langle e_s,N(j)\rangle\cdot dj\, V_s.$$

The analoguous equations hold for c_β with V_s replaced by Y_s^o for each s. Hence

$$c_h\circ c_\beta\, di\, X = \sum_{s'}\sum_s m(j)(Y_s^o,X)\cdot\langle e_s,N(j)\rangle\cdot\langle e_{s'},N(j)\rangle\cdot V_s$$

and

$$c_h\circ c_\beta\, N(j) = \sum_{s'}\sum_s m(j)(V_s,Y_{s'})\cdot\langle e_s,N(j)\rangle\cdot\langle e_{s'},N(j)\rangle\cdot N(j).$$

Therefore

$$\operatorname{tr} c_h\circ c_\beta = 2\cdot\sum m(j)(Y_s^o,V_s)$$

and hence

$$\operatorname{tr}(B_h\circ B_\beta + C_h\circ C_\beta + c_h\circ c_\beta) = 3\sum_s m(j)(Y_s^o,V_s)$$

showing that

$$\int dh\cdot\beta\,\mu(j) = 3\cdot\sum_{s=1}^{n}\int m(j)(Y_s^o,V_s)\mu(j) = 0$$

since Y_s^o and V_s are L_2-orthogonal for all $s = 1,\ldots n$.

An immediate consequence of theorem 1 and 2 is the following:

<u>Corollary 3</u> If F is a $\mathbb{R}^n$-invariant constitutive law of the form $F = d^*\tilde{F}$ and if $\tilde{F}$ admits an integral representation by

$$\alpha\colon E(M,\mathbb{R}^n)/\mathbb{R}^n \longrightarrow A^1(M,\mathbb{R}^n)$$

with dh(dj) referred to as dh(j) the maximal integrable part of $\alpha(dj)$ for all $j \in E(M,\mathbb{R}^n)$, that is if

$\tilde{F}(dj)(dk) = \int dh(j)\cdot dk\,\mu(j) \quad \forall\, j \in E(M,\mathbb{R}^n)$ and $\forall\, k \in C^\infty(M,\mathbb{R})$,

then F admits an integral representation of the form

$F(j)(k) = \int \langle\varphi(j),k\rangle\mu(j) \quad \forall\, j \in E(M,\mathbb{R}^n)$ and $\forall\, k \in C^\infty(M,\mathbb{R}^n)$

with

$$\varphi(j) = - dj(div(B_h(j) + C_h(j)) + 2\, W(j)U_h(i)) \quad (*)$$
$$+ (tr\, B_h(j) \circ W(j)) \cdot N(j).$$

Here $U_h(j) \in \Gamma TM$ is such that $dj\, U_h(j) = c_h(j)N(j)$ and $dh(j)$ is represented as

$$dh(j) = c(j)_h \cdot dj + dj \cdot C_h(j) + dj \cdot B_h(j).$$

Moreover dh is uniquely determined by $\tilde{F}$.

Let us express (*) in corollary 3 more specifically in terms of X_h and Ψ_h. We know (cf. [Bi]) that

$$div_j\, \nabla(j)\, Z = \Delta(j)Z \quad (= - tr\, \nabla(j)^2 Z) \; \forall\, Z \in \Gamma TM.$$

$\Delta(j)$ being the Laplace-Beltrami operator of $m(j)$. Since

$$L_Z\, (m(j))(X,Y) = m(j)((\nabla(j)Z + \nabla(j)Z,Y)$$

we obtain with $\mathbb{L}_Z$ as an abbriviation of $\nabla(j)Z + \tilde{\nabla}(j)Z$ and a moving frame $e_r,\dots,e_{n-1}$ orthonormal with respect to $m(j)$

$$tr\, \mathbb{L}_Z \circ W(j) = \sum_r m(j)(\mathbb{L}_Z \circ W(j)e_r, e_r)$$
$$= \sum_r m(j)(\nabla(j)_{W(j)e_r} Z, e_r) + m(j)(W(j)e_r, \nabla(j)_{e_r} Z)$$
$$= 2 \sum_r m(j)(W(j)\nabla(j)_{e_r} Z, e_r)$$
$$= 2\, (div_j(W(j)Z) - m(j)(div_j W(j), Z)).$$

However (cf. [Bi])

$$div_j(\tau W(j)) = W(j) grad_j \tau + \tau \cdot grad_j H(j) \quad \forall\, \tau \in C^\infty(M, \mathbb{R}^n)$$

and thus

$$m(j)(div_j W(j), X) = dH(j)(X) \quad \forall \quad X \in \Gamma TM$$

$H(j)$ being the trace of the Weingarten map $W(j)$. Obviously

$$U_h(j) = W(j)X_h - grad_j\, \Psi_h(j) \quad \forall\, h \in C^\infty(M, \mathbb{R}^n).$$

Therefore corollary 3 turns into

<u>Corollary 4</u> Let F be an $\mathbb{R}^n$-invariant constitutive law of the form $F = d^*\tilde{F}$ where $\tilde{F}$ is represented by

$$\alpha:\, E(M, \mathbb{R}^n)/\mathbb{R}^n \longrightarrow A^1(M, \mathbb{R}^n).$$

Without loss of generality $\alpha(dj)$ can assumed to be of the form $\alpha(dj) = dh(j)$ with $h(j)$ represented as

$$h(j) = dj\, X_h(j) + \Psi_h(j) \cdot N(j) \quad \forall\, j \in E(M, \mathbb{R}^n)$$

for some $X_h(j) \in \Gamma TM$ and $\Psi_h(j) \in C^\infty(M, \mathbb{R}^n)$. The one form F admits an integral representation

$$F(j)(k) = \int \langle \Psi(j), k \rangle \mu(j) \quad \forall\, j \in E(M, \mathbb{R}^n) \text{ and } \forall\, C^\infty(M, \mathbb{R}^n)$$

with

$$\psi(j) = - dj(\Delta(j)X_h(j) - W(j)\mathrm{grad}_j\, \Psi_h(j) + \Psi_h(j)\ \mathrm{grad}_j\ H(j) + 2\cdot W(j)^2 X_h) + + (\mathrm{div}_j W(j)X_h - dH(j)(X_h) + \Psi_h(j)\ \mathrm{tr}\ W(j)^2)\cdot N(j).$$

The stress tensor of the integrable part of α is hence

$$T(dh(j),j)(X,Y) = m(j)((\nabla(j)X_h(j) + \Psi_h(j)m(j)(W(j)X,Y)$$

for any $X,Y \in FTM$.

Remark: This corollary shows that only the integrable part of the stress form (and not the stress tensor) determines F fully.

Examples:

$F \in A^1(E(M,\mathbb{R}^n),\mathbb{R})$ will assumed to be $\mathbb{R}^n$-invariant that is of the form

$$F = d^*\tilde{F}$$

$\tilde{F}$ is supposed to admit an integral representation by $\alpha \in C^\infty(E(M,\mathbb{R}^n),A^1(M,\mathbb{R}^n))$ of which the integrable part is assumed to be dh.

1) Let $h(j) = j$ for any $j \in E(M,\mathbb{R}^n)$. Hence $dj = dj\cdot id_{TM}$ and therefore

$$\tilde{F}(dj,dk) = \int \mathrm{tr}\ B_{dk}(j)\mu(j).$$

Thus

$$\varphi(j) = (\mathrm{tr}\ W(j))\cdot N(j) = H(j)\cdot N(j)$$

and if

$$V(j) := \int \mu(j), \quad \forall\ j \in E(M,\mathbb{R}^n)$$

we moreover have

$$F(j,k) = DV(j)(k) = \int H(j)\cdot\langle N(j),k\rangle\mu(j).$$

The stress tensor $T(dj,j)$ is obviously $m(j)$ for all $j \in E(M,\mathbb{R}^n)$.

2) Let $h(j) = N(j)$ for any $j \in E(M,\mathbb{R}^n)$. Then $dh = dj\cdot W(j)$ and therefore

$$\varphi(j) = - \mathrm{div}_j\ W(j) + (\mathrm{tr}W(j)^2)\cdot N(j) = - dj\ \mathrm{grad}_j\ H(j) + (\mathrm{tr}\cdot W(j)^2)\cdot N(j).$$

If we use moreover the formula

$$\lambda(j) = -\operatorname{tr} W(j)^2 + (\operatorname{tr} W(j))^2$$
$$= -\operatorname{tr} W(j)^2 + H(j)^2$$

where $\lambda(j)$ is the scalar curvature of $m(j)$ then

$$\varphi(j) = -dj\ \operatorname{grad}_j H(j) - (\lambda(j) - H(j)^2)\cdot N(j) \quad \forall\ j \in E(M,\mathbb{R}^n).$$

The stress tensor $T(djW(j),j)$ is obviously the second fundamental form for any $j \in E(M,\mathbb{R}^n)$.

3) Let $h(j) = dj\ X_h(j)\ \forall\ j \in E(M,\mathbb{R}^n)$. Then

$$B_h(j) = ½\cdot\mathbb{L}_{X_h}$$

and $C_h(j) = ½\cdot(\nabla(j)X_h(j) - \tilde{\nabla}(j)X_h(j))$. Hence for all $j \in E(M,\mathbb{R}^n)$

$$\varphi(j) = -dj(\nabla(j)X_h(j) + 2\ W(j)^2X_h(j)) +$$
$$+ (\operatorname{div} W(j)X_h - dH(j)(X_h))\cdot N(j).$$

The stress tensor at (djX_h, dj) applied to any $X,Y \in \Gamma TM$ is for all $j \in E(M,\mathbb{R}^n)$

$$T(djX_h,j)(X,Y) = m(j)(\nabla(j)_X X_h, Y).$$

4) Let $h(j)$ be such that
$C_h(j) = 0$ and $B_h(j) = ½\ \mathbb{L}_{\nu\cdot X_h(j)}\ \forall\ j \in E(M,\mathbb{R}^n)$ and some $\nu \in \mathbb{R}$.
This means that $X_h(j)$ is a gradient with respect to $m(j)$. Hence

$$h(j) = dj\ \operatorname{grad}_j \tau(j) \qquad \forall\ j \in E(M,\mathbb{R}^n)$$

and some $\tau(j) \in C^\infty(M,\mathbb{R}^n)$. We may then proceed as in (3).

5) Choose α such that

$$\alpha(dj) = ½\ \mathbb{L}_{\nu\cdot Y(j)} \qquad \forall\ j \in E(M,\mathbb{R}^n) \text{ and some } \nu \in \mathbb{R}.$$

where $Y(j) \in \Gamma TM$ for any j. Hence

$$\varphi(j) = -\operatorname{div}_j ½\ \mathbb{L}_{\nu Y(j)} + ½\ (\operatorname{tr} \mathbb{L}_{Y(j)}\circ W(j))\cdot N(j)$$
$$= -dj(\nu\cdot\Delta(j)Y(j) + \nu(R(j)Y(j) + \operatorname{grad}_j \operatorname{div}_j Y(j))$$
$$+ \nu(\operatorname{div}_j W(j)Y(j) - dH(j)(X_h))\cdot N(j)$$

where $m(j)(R(j)X,Y) = \operatorname{Ric}(i)(X,Y)$ with $\operatorname{Ric}(i)$ the Ricci tensor of $m(j)$. We leave it to the reader to determine the integrable part of α.

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