A Note on Positive Supermartingales in Ruin Theory

Klaus D. Schmidt

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Klaus D. SCHMIDT

Seminar für Statistik, Universität Mannheim, A 5,
D-6800 Mannheim, West Germany

In this note we give an elementary proof of Kolmogorov's inequality for positive supermartingales. As an application we obtain a Lundberg type inequality for a class of surplus processes with i.i.d. increments for which an adjustment coefficient need not exist.

Keywords: Adjustment coefficient, Kolmogorov's inequality, Lundberg's inequality, positive supermartingales, ruin theory.

1. Introduction

All random variables considered in this paper are defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. For a subset A of Ω , let χ_A denote its indicator function $\Omega : \longrightarrow \{0,1\}$.

Let G be an integrable random variable. G has an adjustment coefficient if there exists some $R \in (0,\infty)$ satisfying $E[e^{-RG}] = 1$; necessary and sufficient conditions for an adjustment coefficient of G to exist have been given by Mammitzsch (1986).

Consider now $u \in (0,\infty)$, a sequence $\{G_n\}$ of i.i.d. random variables having the same distribution as G , and the surplus process $\{U_n\}$, given by

$$U_n := u + \sum_{k=1}^{n} G_k$$

for all $n \in \mathbb{N}$. If G has an adjustment coefficient, then the probability of ruin satisfies Lundberg's inequality

$$P(\inf_{\mathbb{N}} U_n \leq 0) \leq e^{-Ru} .$$

Gerber (1973,1979) has shown that Lundberg's inequality can be obtained from Kolmogorov's inequality for positive supermartingales. Unfortunately, however, the traditional proofs of Kolmogorov's inequality involve a nontrivial property of supermartingales, and it appears that this fact makes the supermartingale approach appear much less attractive than it is.

In this note we give an entirely elementary proof of Kolmogorov's inequality for positive supermartingales. As an immediate application, we obtain a Lundberg type inequality for a class of surplus processes for which an adjustment coefficient of G need not exist.

2. Kolmogorov's inequality

Let $\{X_n\}$ be a sequence of integrable random variables. For each $n \in \mathbb{N}$, let F_n denote the σ-algebra generated by $\{X_1, \ldots, X_n\}$.
A mapping $\tau : \Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ is

- a stopping time if $\{\tau=n\} \in F_n$ holds for all $n \in \mathbb{N}$, and it is
- bounded if $\sup_\Omega \tau(\omega) < \infty$.

Let T denote the collection of all bounded stopping times for $\{F_n\}$.
For $\tau \in T$, define

$$X_\tau := \sum_{n=1}^{\infty} \chi_{\{\tau=n\}} X_n .$$

Then X_τ is an integrable random variable satisfying

$$E X_\tau = \sum_{n=1}^{\infty} E[\chi_{\{\tau=n\}} X_n] ;$$

note that all sums extend only over a finite number of terms since τ is bounded. The following result is well-known in the theory of asymptotic martingales; see e.g. Gut and Schmidt (1983) and the references given there:

2.1. Lemma. The inequality

$$P(\sup_{\mathbb{N}} |X_n| \geq \varepsilon) \leq \frac{1}{\varepsilon} \sup_T E|X_\tau|$$

holds for all $\varepsilon \in (0,\infty)$.

Proof. Let us assume that the X_n are all positive. For all $n \in \mathbb{N}$, define sets

$$B_n := \{X_n \geq \varepsilon\} \cap \bigcap_{k=1}^{n-1} \{X_k < \varepsilon\}$$

$$C_n := \Omega \setminus \bigcup_{k=1}^{n} B_k$$

and a stopping time $\tau_n \in T$ by letting

$$\{\tau_n = k\} := \begin{cases} B_k & , \text{ if } k \in \{1,\dots,n-1\} \\ B_n \cup C_n & , \text{ if } k = n \end{cases} .$$

Then we have

$$E\left[\sum_{k=1}^{n} \varepsilon \chi_{B_k}\right] \leq E X_{\tau_n} ,$$

hence

$$E\left[\varepsilon \sum_{k=1}^{\infty} \chi_{B_k}\right] \leq \sup_T EX_\tau ,$$

by the monotone convergence theorem, and thus

$$\varepsilon P(\sup_{\mathbb{N}} X_n \geq \varepsilon) \leq \sup_T EX_\tau ,$$

which yields the assertion. □

The sequence $\{X_n\}$ is a <u>supermartingale</u> if $E[\chi_A X_{n+1}] \leq E[\chi_A X_n]$ holds for all $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$.

<u>2.2. Lemma.</u> <u>If</u> $\{X_n\}$ <u>is a supermartingale, then</u> $EX_\tau \leq EX_1$ <u>holds for all</u> $\tau \in T$.

Proof. Choose $n \in \mathbb{N}$ satisfying $\tau \leq n$. Then we have

$$\begin{aligned} E[\chi_{\{\tau=k\}} X_k] + E[\chi_{\{\tau\geq k+1\}} X_{k+1}] & \\ &\leq E[\chi_{\{\tau=k\}} X_k] + E[\chi_{\{\tau\geq k+1\}} X_k] \\ &= E[\chi_{\{\tau\geq k\}} X_k] \end{aligned}$$

for all $k \in \{1,\dots,n\}$, and thus, by induction,

$$EX_\tau = \sum_{k=1}^{n} E[\chi_{\{\tau=k\}} X_k] \leq E[\chi_{\{\tau\geq 1\}} X_1] = EX_1 ,$$

as was to be shown. □

Tho following result is <u>Kolmogorov's inequality</u> for positive supermartingales:

2.3. Theorem. If $\{X_n\}$ is a positive supermartingale, then

$$P(\sup_{\mathbb{N}} X_n \geq \varepsilon) \leq \frac{1}{\varepsilon} EX_1$$

holds for all $\varepsilon \in (0,\infty)$.

This follows from Lemmas 2.1 and 2.2.

We remark that Theorem 2.3 is usually deduced from the nontrivial fact that a positive supermartingale $\{X_n\}$ satisfies $EX_\tau \leq EX_1$ for arbitrary stopping times τ ; see e.g. Neveu (1972).

3. Lundberg's inequality

We now return to the surplus process $\{U_n\}$:

3.1. Theorem. *The inequality*

$$P(\inf_{\mathbb{N}} U_n \leq 0) \leq e^{-\rho u}$$

holds for all $\rho \in (0,\infty)$ *satisfying* $E[e^{-\rho G}] \leq 1$.

Proof. For all $n \in \mathbb{N}$, define

$$X_n := \prod_{k=1}^{n} e^{-\rho G_k} .$$

Then we have

$$E[\chi_A X_{n+1}] = E[\chi_A X_n] E[e^{-\rho G_{n+1}}] \leq E[\chi_A X_n]$$

for all $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$. Therefore, $\{X_n\}$ is a positive supermartingale, and Theorem 2.3 yields

$$\begin{aligned} P(\inf_{\mathbb{N}} U_n \leq 0) &= P(\sup_{\mathbb{N}} \sum_{k=1}^{n} (-G_k) \geq u) \\ &= P(\sup_{\mathbb{N}} X_n \geq e^{\rho u}) \\ &\leq e^{-\rho u} E[e^{-\rho G_1}] \\ &\leq e^{-\rho u} , \end{aligned}$$

as was to be shown. □

Define now

$$I(G) := \{ t \in \mathbb{R} \mid E[e^{tG}] < \infty \}$$

and

$$J(G) := \{ t \in \mathbb{R} \mid E[e^{tG}] < 1 \} .$$

As a consequence of Theorem 3.1 we obtain the following result:

<u>3.2.</u> <u>Corollary.</u> <u>If</u> $\inf I(G) < 0 < EG$, <u>then</u>
$P(\inf_{\mathbb{N}} U_n \leq 0) \leq \inf_{J(G)} e^{tu} < 1$.

Proof. The assumption on G implies the existence of some $t \in (-\infty, 0)$ satisfying $E[e^{tG}] < 1$; see Mammitzsch (1986). The assertion now follows from Theorem 3.1. □

<u>3.3.</u> <u>Corollary.</u> <u>If</u> G <u>has an adjustment coefficient</u> R <u>,</u>
<u>then</u> $P(\inf_{\mathbb{N}} U_n \leq 0) \leq e^{-RG}$.

This follows from Corollary 3.3.

We remark that the hypothesis of Corollary 3.2 does <u>not</u> imply that G has an adjustment coefficient; see Mammitzsch (1986).

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