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Krichever-Novikov Algebras for More than Two Points

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Abstract

Krichever-Novikov algebras of meromorphic vector fields with more than two poles on higher genus Riemann surfaces are introduced. The structure of these algebras and their induced modules of forms of weight λ is studied.

1. Introduction

In the study of conformal field theory a recent generalization of the Virasoro Algebra was given by Krichever-Novikov [1][2] for Riemann surfaces of higher genus. Let X be a fixed Riemann surface of genus g. We choose two "generic " points P_+ and P_- and consider the Lie algebra of meromorphic vector fields on X which are holomorphic on $X \setminus \{P_+, P_-\}$. Krichever and Novikov showed that this algebra admits a central extension. They constructed a series of representation coming from the operation of the vector fields on the meromorphic forms of weight λ which are holomorphic on $X \setminus \{P_+, P_-\}$. This algebra (with or without central extension) is now usually called Krichever-Novikov algebra (short: KN algebra). For $g = 0$, hence $X = \mathbb{P}^1$ (all varieties are over the complex numbers $\mathbb{C}$) this algebra is exactly the Virasoro algebra. A crucial part in the theory is the fixing of a distinguished basis of the vector fields and of the forms, the so called KN basis.

There is a meromorphic (1-) differential ρ holomorphic on $X \setminus \{P_+, P_-\}$ with residue $+1$ at P_+ and residue -1 at P_- and only imaginary periods (see [3,p116]). It was pointed out [2][6] that in string theory the level lines of the function (Q denotes a point different from P_+ and P_-)

$$u(p) = \mathbf{Re} \int_Q^P \rho$$

on $X \setminus \{P_+, P_-\}$ could be interpreted as closed string configuration at the proper time τ. Here τ gives the value of the function u along the level line. As $\tau \to \pm\infty$ the level lines become circles around the points $P_\mp$ representing free incoming and outgoing strings.

Hence it is quite natural to ask for more strings coming together and interacting. In terms of KN algebras this is the question of the structure of the algebra of meromorphic vector fields which are holomorphic except at a certain points.

The aim of this paper is to give the results of such a generalization. Details of the calculation and further developments will be found in a separate publication [4]. For the generalities on Riemann surfaces , the theorem of Riemann-Roch and its application to the KN algebra for two points see [3]. For the interpretation in physics see [6-14].

2. Fundamental definitions

Let X be a Riemann surface of genus g. We always assume X to be compact and without boundary. X can also be considered as a connected, projective, nonsingular curve over $\mathbb{C}$. A formal sum of points $Q_1, \ldots, Q_l \in X$ with integer multiplicities

$$D = \sum_{i=1}^{l} n_i Q_i \qquad (1)$$

is called a divisor. K denotes the canonical divisor, resp. the canonical divisor class, resp. the canonical line bundle (which is the same as the holomorphic cotangent bundle). The sections of the canonical line bundle are the differentials. $K^\lambda := K^{\otimes\lambda}$ is λ times the tensor product of the bundle K with itself. For $\lambda < 0$ this means $K^\lambda := (K^*)^{\otimes|\lambda|}$. K^* denotes the dual bundle, the holomorphic tangent bundle. If we use the divisor notation, then K^λ corresponds to the divisor $\lambda \cdot K$.
The definition of K^λ with $\lambda \in \mathbb{Z}$ is clear. To define it for λ half integer we have to choose a "square root" of K. This is not unique for $g \geq 1$. For arbitrary λ we have to consider coverings of X. In this paper we will restrict ourselves to integer valued λ. The case of arbitrary λ will be covered in [4].

The (meromorphic) sections of K^λ are called (meromorphic) forms of weight λ. Forms of weight -1 are the vector fields. Let D be a divisor as above then $H^0(X, \lambda K + D)$ is the vector space of meromorphic λ - forms which are holomorphic outside the support of D (which is the set of points appearing with nonvanishing multiplicities) and have a zero of at least order $-n_i$ at the point Q_i. As usual a zero of negative order m_i is interpreted as a pole of order $-m_i$. We use the notation $\operatorname{ord}_{Q_i}(s)$ for the order of the zero of the section s at the point Q_i.

By the theorem of Riemann-Roch (see [3,p.107]) and using $\deg(\lambda K) = 2\lambda(g-1)$ we get

$$
\begin{aligned}
(2) \qquad \dim H^0(X, \lambda K + D) - \dim H^0(X, (1-\lambda)K - D) &= \deg(\lambda K + D) - g + 1 \\
&= (2\lambda - 1)(g-1) + \deg D.
\end{aligned}
$$

Let k be a natural number bigger or equal to two. Let $P_1, P_2, \ldots, P_k$ be k different "generic" points of X. In our context generic means that there is a countable set of points of X which have to be avoided. We choose an arbitrary numerical order of the points. This we will keep fixed. In addition we fix local coordinates z_l around the points P_l with $z_l(P_l) = 0$ for $l = 1, \ldots, k$. X^0 denotes $X \setminus \{P_1, P_2, \ldots, P_k\}$.

3. The level lines

LEMMA 1. *Let X be a Riemann surface of genus g, $P_1, P_2, \ldots, P_k$ k different points $(k \geq 2)$. Let*

$$
(3) \qquad c_i \in \mathbb{C} \quad , i = 1, \ldots, k \quad \text{with} \quad \sum_{i=1}^{k} c_i = 0 ,
$$

then there exists a unique meromorphic differential ρ on X such that

(a) ρ is holomorphic on $X \setminus \{P_1, P_2, \ldots, P_k\}$,

(b) $\text{res}_{P_i}(\rho) = c_i$, for $i = 1, \ldots, k$,

(c) *ρ has only imaginary periods.*

The proof is a generalization of the proof in [**3**,p.116], see also [**4**].

Let us apply the lemma in the following situation. Let $1 \le l < k$ be a number, $P_1, \ldots, P_l$ and $P_{l+1}, \ldots, P_k$ be a partition of the points $P_1, P_2, \ldots, P_k$. If we set

$$(4) \qquad c_i = \frac{1}{2l}, \quad i = 1, \ldots, l \quad \text{and} \quad c_i = -\frac{1}{2(k-l)}, \quad i = l+1, \ldots, k$$

we get $\sum_i c_i = 0$. Let ρ be the differential existing by lemma 1. We choose $Q \in X$ with $Q \ne P_i$, $i = 1, \ldots, k$.

$$u(p) := \mathbf{Re} \int_Q^P \rho$$

is now a harmonic function defined on $X^0 = X \setminus \{P_1, P_2, \ldots, P_k\}$ with

$$(5) \quad \lim_{P \to P_i} u(P) = -\infty, \quad i = 1, \ldots, l \quad \text{and} \quad \lim_{P \to P_i} u(P) = \infty, \quad i = l+1, \ldots, k\ .$$

The level line for $\tau \in \mathbb{R}$ is defined as

$$(6) \qquad C_\tau := \{P \in X^0 \mid u(P) = \tau\}\ .$$

Varying τ defines a global fibration of the surface $X \setminus \{P_1, P_2, \ldots, P_k\}$. Each level line splits in a union of disjoint (real) curves. Singular points can only occur at the zeros of the differential ρ. For $\tau \ll 0$ the level line C_τ splits in a collection of l components C^r, $r = 1, \ldots, l$, where each component C^r is a curve diffeomorphic to S^1 around the point P_r. For $\tau \gg 0$ we get the same situation around the points P_r, $(l+1) \le r \le k$.

Finally, if ω is a meromorphic differential which is holomorphic on X^0, then the value of the integral of ω along any nonsingular level line will be the same.

4. The KN - algebra and the KN - modules

DEFINITION.

(a) The (generalized) Krichever-Novikov algebra (KN algebra) of the Riemann surface X of genus g and the points $(k \ge 2)$ $P_1, P_2, \ldots, P_k$ is the Lie algebra of meromorphic vector fields on X, which are holomorphic on $X^0 = X \setminus \{P_1, P_2, \ldots, P_k\}$. It is denoted by $\mathrm{KN}(P_1, P_2, \ldots, P_k)$ or just by KN_k.

(b) The (generalized) Krichever-Novikov module (KN module) of weight $\lambda \in \mathbb{Z}$ is the vector space of meromorphic forms of weight λ which are holomorphic on $X^0 := X \setminus \{P_1, P_2, \ldots, P_k\}$. It is denoted by $F^\lambda(P_1, P_2, \ldots, P_k)$, or just by F_k^λ.

Of course, $\mathrm{KN}(P_1, P_2, \ldots, P_k)$ equals $F^{-1}(P_1, P_2, \ldots, P_k)$ as vector space.

By taking the Lie derivative F_k^λ is a Lie algebra module over KN_k. In local terms the action can be described as follows. Let e be a meromorphic vector field and f be a meromorphic form of weight λ. Locally they are given as

$$e(z)_| = \alpha(z)\frac{\partial}{\partial z}, \quad f(z)_| = \beta(z)\, dz^\lambda, \quad \text{with } dz^\lambda := (dz)^{\otimes \lambda} \ . \tag{7}$$

Then the Lie derivative is given by

$$e\,.\,f(z)_| := L_e(f)\,(z)_| = \left(\alpha(z)\frac{\partial \beta(z)}{\partial z} + \lambda \cdot \beta(z)\frac{\partial \alpha(z)}{\partial z}\right) dz^\lambda \ . \tag{8}$$

5. A basis for F_k^λ in the cases $(g \geq 2, \lambda \neq 0, 1)$ and $(g = 0, \lambda \in \mathbb{Z})$

First we will give a set of generators which is completely symmetric in the points $P_l, l = 1, \ldots, k$. We set

$$M(\lambda) = (2\lambda - 1)(g - 1) - 1 \ . \tag{9}$$

PROPOSITION 1. *Let* $n_1, n_2, \ldots, n_{k-1} \in \mathbb{Z}$ *and* $n_k = M(\lambda) - \sum_{i=1}^{k-1} n_i$ *then*

$$H^0(X, \lambda \cdot K - n_1 P_1 - n_2 P_2 \cdots - n_k P_k)$$

is a one-dimensional vector space. It is generated by a (up to multiplication with a constant) unique form

$$f = f^\lambda(n_1, n_2, \ldots, n_k) \quad \text{with} \quad \mathrm{ord}_{P_i}(f) = n_i, \ i = 1, \ldots, k. \tag{10}$$

For the details of the calculation we refer to [**4**]. The argument is similar to the case considered in [**2**]. Let me scetch the main idea. Starting point is the following

LEMMA 2. *Let $\mathcal{L}$ be a line bundle, L its corresponding divisor, n a natural number. Set $l := \dim H^0(X, L)$, then*

$$\dim H^0(X, L - nP) = \max(l - n, 0) \tag{11}$$

if P is a generic point on X.

We use the fact, that in our cases of λ either

$$\begin{aligned} \deg(\lambda \cdot K) &= \lambda(2g - 2) \geq 2g - 1 \qquad \text{or} \\ \deg((1 - \lambda) \cdot K) &= (1 - \lambda)(2g - 2) \geq 2g - 1 \ . \end{aligned} \tag{12}$$

Hence one of this divisors is nonspecial, which says that the second term of the left hand side of Riemann-Roch theorem vanishes [3,p.107]. With Riemann-Roch we calculate

$$\begin{aligned} \dim H^0(X,\lambda\cdot K) &= (2\lambda-1)(g-1) \qquad \text{or} \\ \dim H^0(X,(1-\lambda)\cdot K) &= (1-2\lambda)(g-1)\ . \end{aligned} \tag{13}$$

Using (11) and assuming $P_1, P_2, \ldots, P_k$ to be generic we calculate

$$\dim H^0(X, \lambda\cdot K - n_1P_1 - n_2P_2 \cdots - n_kP_k) = 1, \tag{14}$$

$$\dim H^0(X, \lambda\cdot K - n_1P_1 - n_2P_2 \cdots - (n_k+a)P_k) = 0,\ a \in \mathbb{N}\ . \tag{15}$$

The former implies the statement on the dimension, the latter implies that the generator has exactly the prescribed order at the points.

By requiring for the local representation at the point P_k

$$f_|(z) = z_k^{n_k}(1+O(z_k))\, dz^\lambda \tag{16}$$

the $f = f^\lambda(n_1, n_2, \ldots, n_k)$ is completely fixed. We will assume this normalization in the following.

PROPOSITION 2. *The set*

$$\{\ f^\lambda(n_1,n_2,\ldots,n_k)\ |\ n_1,n_2,\ldots,n_{k-1} \in \mathbb{Z},\ n_k = M(\lambda) - \sum_{i=1}^{k-1} n_i\ \} \tag{17}$$

is a set of generators for $F^\lambda(P_1, P_2, \ldots, P_k)$.

For $k > 2$ this set is not linearly independent. To get a basis of F_k^λ we introduce the following types of generators $(n, l \in \mathbb{Z})$

$$f_n(\lambda) := f^\lambda(n,0,\ldots,0,M(\lambda)-n), \quad n \geq 0 \qquad \text{type I} \tag{18}$$

$$f_n(\lambda) := f^\lambda(n,0,\ldots,0,M(\lambda)-n), \quad n < 0 \qquad \text{type II} \tag{19}$$

$$f_n^l(\lambda) := f^\lambda(0,\ldots,n,\ldots,0,M(\lambda)-n), \quad n < 0 \qquad \text{type III}_l \tag{20}$$

In the definition (20) $l = 2, \ldots, k-1$ and the number n has to be plugged into the l-th position. If it is convenient we will also use the notation $f_n^1(\lambda)$ to denote generators of type I and type II.

PROPOSITION 3. *The set of*

(21) $\qquad f_n(\lambda),\ n \in \mathbb{Z}$ and $\qquad f_n^l(\lambda),\quad n \in \mathbb{Z}, n \le -1,$ with $\ 2 \le l \le k-1$

is a basis of $F^\lambda(P_1, P_2, \ldots, P_k)$.

If there is no ambiguity we will drop the λ in the notation. Due to the special importance of certain weights we introduce

$$e_n^l := f_n^l(-1), \qquad \text{and} \qquad \Omega_n^l := f_n^l(2)$$

and (assuming the result of the next section already)

$$A_n^l := f_n^l(0), \qquad \text{and} \qquad \omega_n^l := f_n^l(1)\ .$$

Of course, it is also possible to embed the Riemann surface into its Jacobian and to describe in some sense more explicitly the above forms in terms of theta functions, prime forms etc. as it was done for $k = 2$ in [**11**] and [**14**]. This will be covered in [**4**].

6. A basis for F_k^λ in the cases $(g \ge 2, \lambda = 0$ or $\lambda = 1)$ and $(g = 1, \lambda \in \mathbb{Z})$

Due to the fact that $\lambda \cdot K$ is a special divisor we have to modify the argument and the basis.

PROPOSITION 4. $(g \ge 2, \lambda = 1)$. *A set of symmetric generators of* F_k^1 *is given by* $(n_i \in \mathbb{Z},\ i = 1, \ldots, k-1)$

$$f^1(n_1, n_2, \ldots, n_{k-1}, (g-1) - \sum_{i=1}^{k-1} n_i), \quad n_i \ge 0,\ \sum_{i=1}^{k-1} n_i \le (g-1), \tag{22}$$

$$f^1(n_1, n_2, \ldots, n_{k-1}, (g-2) - \sum_{i=1}^{k-1} n_i), \quad n_i \ne -1, \tag{23}$$

where at least one $n_i \le -2$ *or* $\sum_{i=1}^{k-1} n_i \ge g$ *and*

$$\begin{gathered} f^1(-1,-1,0,\ldots,0),\ f^1(-1,0,-1,\ldots,0),\ \ldots, \\ f^1(0,-1,-1,\ldots,0),\ \ldots, f^1(0,\ldots,0,-1,-1)\ . \end{gathered} \tag{24}$$

The elements f in case (22) and (23) are uniquely given by a similar argument as above. In case (24) we understand by $f^1(0,0,\ldots,0,-1,\ldots,-1,\ldots,0)$ the unique differential given by lemma 1 with $c_l = -1$ and $c_m = 1$. Here l and m are the entries being equal to -1 in the description of f with $l < m$. Here again we have similar types $(n \in \mathbb{Z})$

$$\omega_n := f_n(1) := f^1(n, 0, \ldots, 0, \begin{cases} g-2-n, & n \ge g \\ g-1-n, & 0 \le n \le g-1 \end{cases}), \qquad \text{type I} \tag{25}$$

$$(26) \qquad \omega_n := f_n(1) := \begin{cases} f^1(-1,0,\ldots,0,-1), & n=-1 \\ f^1(n,0,\ldots,0,g-2-n), & n\le -2 \end{cases} \qquad \text{type II}$$

$$(27) \qquad \omega_n^l := f_n^l(1) := \begin{cases} f^1(0,\ldots,-1,\ldots,0,-1), & n=-1 \\ f^1(0,\ldots,n,\ldots,0,g-2-n), & n\le -2 \end{cases} \qquad \text{type III}_l$$

In (27) we have $l=2,\ldots,k-1$.

PROPOSITION 5. *The set of the generators (25)-(27) is a basis for* $F^{-1}(P_1,P_2,\ldots,P_k)$.

In the case of genus 1 K is the trivial bundle. Hence all tensor powers are again trivial. For this reason a basis of F_k^0 is a basis for all F_k^λ.

PROPOSITION 6. *($g\ge 2, \lambda=0$ or $g=1, \lambda\in\mathbb{Z}$). A set of symmetric generators is given by ($n_i\in\mathbb{Z},\ i=1,\ldots,k-1$)*

$$(28) \qquad f^0(n_1,n_2,\ldots,n_{k-1},-g-\sum_{i=1}^{k-1} n_i), \quad \text{at least one } n_i>0, \text{ or } \sum_{i=1}^{k-1} n_i < -g,$$

$$(29) \qquad f^0(n_1,n_2,\ldots,n_{k-1},-g-1-\sum_{i=1}^{k-1} n_i), \ n_i\le 0, \ \sum_{i=1}^{k-1} n_i \ge -g-1,$$

$$(30) \qquad f^0(0,0,0,\ldots,0)\equiv 1$$

In case (28) uniqueness is again by a similar argument as above. In case (29) such a f is only unique up to muliplication with a constant and adding a constant. Hence normalization at the point P_k does not fix it completely. We will give a method for fixing it by duality later on. The types are ($n\in\mathbb{Z}$)

$$(31) \quad A_n := f_n(0) := \begin{cases} f^0(n,0,\ldots,0,-g-n), & n>0 \\ f^0(0,0,0,\ldots,0)\equiv 1, & n=0 \end{cases} \qquad \text{type I}$$

$$(32) \quad A_n := f_n(0) := \begin{cases} f^0(n,0,\ldots,0,-g-1-n), & -g\le n<0 \\ f^0(n,0,\ldots,0,-g-n), & n\le -(g+1) \end{cases} \qquad \text{type II}$$

$$(33) \quad A_n^l := f_n^l(0) := \begin{cases} f^0(0,\ldots,n,\ldots,0,-g-1-n), & -g\le n<0 \\ f^0(0,\ldots,n,\ldots,0,-g-n), & n\le -(g+1) \end{cases} \qquad \text{type III}_l$$

with $l=2,\ldots,k-1$.

PROPOSITION 7. *The set of the generators (31)-(33) is a basis for* $F^0(P_1,P_2,\ldots,P_k)$.

7. The structure constants

If we choose the basis in $\mathrm{KN}(P_1, P_2, \ldots, P_k)$ and the basis in $F^\lambda(P_1, P_2, \ldots, P_k)$ as above then the module structure of F_k^λ over KN_k, resp. the structure of the algebra KN_k itself for $\lambda = -1$, is completely fixed by the structure constants $C_{\alpha,\beta}^\gamma \in \mathbb{C}$, given by

$$e_\alpha \cdot f_\beta = \sum_\gamma{}' C_{\alpha,\beta}^\gamma f_\gamma \ . \tag{34}$$

Here α, β, γ are generalized indices. The structure constants are depending on the weight λ . $\sum'$ denotes that only finitely many f_γ actually occur.

By doing local calculations at the points $P_1, P_2, \ldots, P_k$ (similar to [3,p.115]) we can determine which indices γ occur. We restrict ourselves to $g \geq 2$ and $\lambda \neq 0, 1$ or $g = 0$ and $\lambda \in \mathbb{Z}$. In the remaining cases there are some minor modifications due to the exceptional elements in the basis (see [4]).

Let us fix λ. Depending on the type of the basis elements we get the following result:

(type I, type I)

$$e_n \cdot f_m = \sum_{r=\max(0,n+m-1)}^{n+m-1+3g} A_{n,m}^r(\lambda) f_r \ . \tag{35}$$

(type I,type II) and vice versa

$$e_n \cdot f_m = \sum_{r=0}^{n+m-1+3g} A_{n,m}^r(\lambda) f_r + \sum_{r=n+m-1}^{\min(-1,n+m-1+3g)} A_{n,m}^r(\lambda) f_r^1 \ . \tag{36}$$

Of course one of the sum could be the empty sum. This coefficients are exactly the coefficients of the KN algebra (modules) of the two points P_1 and P_k. In the boundary cases we get

$$A_{n,m}^{n+m-1+3g}(\lambda) = -(m + \lambda n) - g(1 + \lambda), \quad A_{n,m}^{n+m-1}(\lambda) = (m + \lambda n) \frac{a_n b_m}{b_{n+m-1}}. \tag{36a}$$

Here a_n is the leading coefficient of e_n at the point P_1 and b_m the leading coefficient of f_m at the point P_1.

(type I,type III_l) and vice versa

$$e_n \cdot f_m^l = \sum_{r=0}^{n+m-1+3g} A_{n,m}^{r,l}(\lambda) f_r + \sum_{r=m-1}^{\min(-1,n+m-1+3g)} A_{n,m}^{r,l}(\lambda) f_r^l \tag{37}$$

(38) $$e^l_n \cdot f_m = \sum_{r=0}^{n+m-1+3g} B^{r,l}_{n,m}(\lambda) f_r + \sum_{r=n-1}^{\min(-1,n+m-1+3g)} B^{r,l}_{n,m}(\lambda) f^l_r$$

The basic method for calculation of the coefficient, is to write e_n, resp. f_m as linear combination of the basis of the KN algebra $\mathrm{KN}(P_l, P_k)$, resp. KN module, do the calculation as above and transfer the result back in terms of our basis. For a detailed calculation see [4]. As can be seen in (38) there is always a term f^l_{-n-1} even if m was chosen to be a big positive number.

(type III_l, type III_l)

(39) $$e^l_n \cdot f^l_m = \sum_{r=0}^{n+m-1+3g} C^{r,l}_{n,m}(\lambda) f_r + \sum_{r=n+m-1}^{\min(-1,n+m-1+3g)} C^{r,l}_{n,m}(\lambda) f^l_r .$$

(type III_l, type III_h), $h \neq l$ or (type II, type III_l):

(40) $$e^l_n \cdot f^h_m = \sum_{r=n-1}^{-1} D^{r,l,h}_{n,m}(\lambda) f^l_r + \sum_{r=m-1}^{-1} E^{r,l,h}_{n,m}(\lambda) f^h_r + \sum_{r=0}^{n+m-1+3g} F^{r,l,h}_{n,m}(\lambda) f_r .$$

If $n + m \leq -3g$ the last sum will not appear. Nevertheless in the first and second sum all terms will occur up to $r = -1$.

For $k = 2$ this is exactly the result of Krichever and Novikov [1]. In their rule of indexing the above looks like

(41) $$e_{(i)} \cdot f_{(j)} = \sum_{l=-g_0}^{g_0} R^l_{i,j} f_{(i+j+l)}, \quad g_0 = \frac{3}{2}g.$$

Both rules of indexing are related by

(42) $$i = n + \frac{3}{2}g - 1 \quad \text{and} \quad j = m + \frac{g}{2} - \lambda(g-1) .$$

Here the structure looks very symmetric. This symmetry does not occur for $k > 2$. Hence we decided not to choose this rule of indexing.

8. Example

To illustrate the result we calculate in the case of $X = \mathbb{P}^1$ (hence $g = 0$) the algebra $\mathrm{KN}(P_1, P_2, \ldots, P_k)$. We choose such a parametrization z of X that P_1 corresponds to $z = 0$ and P_k corresponds to $z = \infty$. The points P_l correspond to $a_l \in \mathbb{C}$ for $l = 2, \ldots, k-1$. In this parametrization it is easy to give the basis in an explicit way

$$
(43) \qquad
\begin{aligned}
e_n &= z^n \frac{\partial}{\partial z}, \quad n \geq 0 \qquad \text{type I} \\
e_n &= z^n \frac{\partial}{\partial z}, \quad n < 0 \qquad \text{type II} \\
e_n^l &= (z - a_l)^n \frac{\partial}{\partial z}, \quad n < 0 \qquad \text{type III}_l \,.
\end{aligned}
$$

Here we normalized (contrary to the general rule) in such a way that the leading coefficient at ∞ is equal to -1.

By direct calculation we see:

$$
(44) \qquad [e_n, e_m] = (m - n) e_{n+m-1}, \quad n, m \in \mathbb{Z}
$$

$$
(45) \qquad [e_n, e_m^l] = \sum_{r=0}^{m+n-1} A_{n,m}^{r,l} e_r + \sum_{r=m-1}^{\min(-1, m+n-1)} A_{n,m}^{r,l} e_r^l \qquad n \geq 0
$$

with $l = 2, \ldots, k-1$ and

$$
(46) \quad A_{n,m}^{r,l} = \left(\sum_{t=r}^{m+n-1} (-1)^{t+r} \binom{t}{r} \binom{n}{t - m + 1} (2m - 1 - t) \right) a_l^{n+m-1-r}, \quad r \geq 0
$$

$$
(46a) \quad A_{n,m}^{r,l} = \binom{n}{r - m + 1} \cdot a_l^{n+m-1-r} \cdot (2m - 1 - r), \qquad r < 0
$$

Furthermore for $l = 2, \ldots, k-1$

$$
(47) \qquad [e_n^l, e_m^l] = (m - n) e_{n+m-1}^l, \quad n, m < 0 \,.
$$

In the remaining case $1 \leq r, l \leq k-1, r \neq l$ we get

$$
(48) \qquad [e_n^r, e_m^l] = \sum_{s=n-1}^{-1} D_{n,m}^{s,r,l} e_s^r + \sum_{s=m-1}^{-1} E_{n,m}^{s,r,l} e_s^l
$$

with (setting $q = m + n - 1$ and $a_1 = 0$)

$$
(49) \qquad D_{n,m}^{s,r,l} = \binom{m}{s - n + 1} (a_r - a_l)^{q-s} (s - 2n + 1)
$$

$$
(50) \qquad E_{n,m}^{s,r,l} = -D_{m,n}^{s,l,r} \,.
$$

In particular, we see that e_{-1}^r and e_{-1}^l will always occur.

9. Duality

If we multiply a form of weight λ and a form of weight $1-\lambda$ we get a form of weight 1, hence a 1-differential. By taking the integral over suitable curves we get a kind of a duality pairing between the above forms. We choose a differential (see equation (26))

$$\rho = \frac{1}{k-1}\sum_{l=1}^{k-1}\omega^l_{-1}. \tag{51}$$

It has residue $-1/(k-1)$ at the points $P_1,\ldots,P_{k-1}$ and residue $+1$ at the point P_k. The level line is a circle around P_k for $\tau \ll 0$ and a collection of circles around the other points for $\tau \gg 0$. Let us denote such a circle (with orientation) around P_l by C^l. For smooth C_τ and for $\omega \in F^1(P_1,P_2,\ldots,P_k)$ we get

$$\sum_{l=1}^{k-1}\frac{1}{2\mathrm{i}\pi}\oint_{C^l}\omega = \frac{1}{2\mathrm{i}\pi}\oint_{C_\tau}\omega = \frac{1}{2\mathrm{i}\pi}\oint_{C^k}\omega. \tag{52}$$

Of course, we could have chosen arbitrary circles C^l around the points P_l without changing the value of the integral.

PROPOSITION 8. *Let $v \in F^{1-\lambda}(P_1,P_2,\ldots,P_k)$ have the representation*

$$v = \sum_{n\ge 0}{}' r_n f_n(1-\lambda) + \sum_{n<0}{}' r_n f_n(1-\lambda) + \sum_{l=2}^{k-1}\sum_{n<0}{}' s^l_n f^l_n(1-\lambda)\ . \tag{53}$$

Then the coefficients can be calculated by

$$r_n = \frac{1}{2\mathrm{i}\pi}\oint_{C^k} f_{-n-1}(\lambda)\cdot v = \frac{1}{2\mathrm{i}\pi}\oint_{C_\tau} f_{-n-1}(\lambda)\cdot v, \qquad n\ge 0 \tag{54}$$

$$r_n = \frac{1}{2\mathrm{i}\pi}\oint_{C^1} f_{-n-1}(\lambda)\cdot v \qquad n<0 \tag{55}$$

$$s^l_{-1} = \frac{1}{2\mathrm{i}\pi}\oint_{C^l} f_0(\lambda)\cdot v \tag{56}$$

$$s^l_{-2} = \frac{1}{2\mathrm{i}\pi}\oint_{C^l} f_1(\lambda)\cdot v - s^l_{-1}\alpha^l_{1,-1}(\lambda)$$

$$\ldots$$

$$s^l_{-r} = \frac{1}{2\mathrm{i}\pi}\oint_{C^l} f_{r-1}(\lambda)\cdot v - \sum_{p=1}^{r-1} s^l_{-p}\alpha^l_{r-1,-p}(\lambda)\ .$$

Here it is defined $(r \geq 0, t < 0)$

$$\alpha^l_{r,t}(\lambda) = \frac{1}{2i\pi}\oint_{C^l} f_r(\lambda)\cdot f^l_t(1-\lambda)\ . \tag{57}$$

We get

$$\alpha^l_{n,-n-1}(\lambda) = 1 \quad \text{and} \quad \alpha^l_{n,m}(\lambda) = 0 \quad \text{if } m < -n-1\ . \tag{58}$$

In the case of $k = 2$ there are no terms of the third kind. Hence we get the (well known) result

$$\frac{1}{2i\pi}\oint_{C_r} f_n(\lambda)\cdot f_m(1-\lambda) = \begin{cases} 1 & m = -n-1 \\ 0 & m \neq -n-1 \end{cases}. \tag{59}$$

In case of genus g equal to 1 or in case of $g \geq 2$ and $\lambda = 0$ or 1 we have to add certain constants to the elements A^l_n for $-g \leq n \leq -1$ (33) to make the proposition 8 also true in this case. But remember, up to now we were only able to fix exactly this elements up to an addition of a constant (see proposition 6). With the duality requirement they are completely determined. See [4] for details.

10. The central extension

Starting from our Krichever-Novikov algebra $\mathrm{KN} = \mathrm{KN}(P_1, P_2, \ldots, P_k)$ we consider central extensions $\widehat{\mathrm{KN}}$ of it. Let E^l_n denote a fixed lift to $\widehat{\mathrm{KN}}$ of the basis element e^l_n of KN. Then $\widehat{\mathrm{KN}}$ is generated by a central element t and the set of E^l_n (n and l as in equation (21)). We get

$$[E_\alpha, t] = 0 \tag{60}$$

$$[E_\alpha, E_\beta] = \sum_\gamma C^\gamma_{\alpha,\beta} E_\gamma + \chi(e_\alpha, e_\beta)t. \tag{61}$$

Here α, β, γ are generalized indices (i.e. $E_\alpha = E^l_n$), $C^\gamma_{\alpha,\beta}$ are the structure constants of the algebra KN

$$[e_\alpha, e_\beta] = \sum_\gamma C^\gamma_{\alpha,\beta} e_\gamma \tag{62}$$

and $\chi(\alpha,\beta) \in \mathbb{C}$ is a 2-cocycle. It is defined for every pair of vector fields and it fulfills

$$\chi(\alpha,\beta) = -\chi(\beta,\alpha) \tag{63}$$

$$\chi([f,g],h) + \chi([g,h],f) + \chi([h,f],g) = 0\ . \tag{64}$$

These conditions are sufficient and necessary for $\widehat{\mathrm{KN}}$ being a Lie algebra.

To construct central extensions we use the method of [2].

DEFINITION. *Let* (U_α, z_α) *be a covering of* X *by coordinate patches and let* $z_\beta = f_{\alpha\beta}(z_\alpha)$ *be the transition functions for non-empty* $U_\alpha \cap U_\beta$. *A meromorphic projective connection is a collection of local meromorphic functions* $R_\alpha(z_\alpha)$ *which are related on nonempty* $U_\alpha \cap U_\beta$ *by*

$$R_\beta(z_\beta)\left(\frac{\partial z_\beta}{\partial z_\alpha}\right)^2 = R_\alpha(z_\alpha) + S(f_{\alpha\beta}) \; . \tag{65}$$

Here $S(h)$ *is the Schwartzian derivative. It is defined as*

$$S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 \; . \tag{66}$$

The $'$ *denotes derivation with respect to the local variable* z.

If all local functions are holomorphic we call R a holomorphic projective connection. There exists always a holomorphic projective connection [**15**,p.202]. Due to the transformation law (65) the difference of two projective connections is always a quadratic differential (i.e. a form of weight 2). Hence by fixing one holomorphic connection R_0 we can get all of them by adding forms of weight 2. In the following we are mainly interested in meromorphic ones which are holomorphic on $X \setminus \{P_1, P_2, \ldots, P_k\}$ and have a pole of at most order 1 at the points P_l, $l = 1, \ldots, k$. We get for $g \geq 2$

$$R = R_0 + \sum_{n=0}^{3g-3} c_n \Omega_n + \sum_{l=1}^{k-1} c_{-1}^l \Omega_{-1}^l, \quad c_n, c_{-1}^l \in \mathbb{C} \tag{67}$$

resp. for $g = 1$ $(\Omega_n^l = A_n^l)$

$$R = R_0 + c_0 \Omega_0 + \sum_{l=1}^{k-1} c_{-1}^l \Omega_{-1}^l, \quad c_0, c_{-1}^l \in \mathbb{C} \; . \tag{68}$$

For $g = 0$ there is only something to add to R_0 if $k \geq 4$

$$R = R_0 + \sum_{s=1}^{k-3} c_s \Omega^s, \quad c_s \in \mathbb{C} \; . \tag{69}$$

The Ω^s are certain forms of weight 2. If we use "standard coordinates" for $g = 1$ and $g = 0$ we can use $R_0 \equiv 0$ in these cases.

With the help of these projective connections we set for vector fields e, h with the local representations

$$e_| = f(z)\frac{\partial}{\partial z}, \quad h_| = g(z)\frac{\partial}{\partial z} \tag{70}$$

$$\bar{\chi}(e,h) := \left(\frac{1}{2}(f'''g - fg''') - R \cdot (f'g - fg')\right) dz \ . \tag{71}$$

This is a meromorphic 1-form which we can integrate along the level lines C_τ according to the differential ρ of equation (51). We set with $\hat{c} \in \mathbb{C}$ an arbitrary constant

$$\chi(e,h) = \frac{\hat{c}}{24\pi \mathrm{i}} \oint_{C_\tau} \bar{\chi}(e,h) = \frac{\hat{c}}{24\pi \mathrm{i}} \oint_{C^k} \bar{\chi}(e,h) \ . \tag{72}$$

PROPOSITION 9. *$\chi(e,h)$ defines a 2-cocycle, hence a central extension of the KN algebra.*

11. Semiinfinite wedge representations

We fix a weight λ. Let F be $F^\lambda(P_1, P_2, \ldots, P_k)$, f_n^l the basis of F, e_n^l the basis of $\mathrm{KN}(P_1, P_2, \ldots, P_k)$. We want to give the elements of the basis a $\mathbb{Z}$ · graduation

$$f_{(i)} := f_n, \quad n \geq 0 \tag{73}$$

$$f_{(i)} := f_n^l, \quad n < 0, l = 1, \ldots, k-1 \text{ where } i = (n+1)(k-1) - l \ . \tag{74}$$

A semiinfinite form is an element of the vector space

$$H = H^\lambda(P_1, P_2, \ldots, P_k)$$

generated by the formal elements

$$w = f_{(i_r)} \wedge f_{(i_{r+1})} \wedge \ldots f_{(i_s)} \wedge f_{(m)} \wedge f_{(m+1)} \wedge \ldots \tag{75}$$

with

$$i_r < i_{r+1} < \ldots < i_s < m \ . \tag{76}$$

The dots in the right part of (75) means that starting from an arbitrary index m all elements with index $k \geq m$ appear.

We have

$$e_{(i)} \cdot f_{(j)} = \sum_{(l)}{}' G_{i,j}^l f_{(l)}, \quad G_{i,j}^l \in \mathbb{C} \ . \tag{77}$$

This action of KN on F we want to transfer onto the vector space H. We try the following naive definition (Leibniz rule)

$$\begin{aligned} e \cdot w :=& (e \cdot f_{(i_1)}) \wedge f_{(i_2)} \wedge \ldots \ + \ f_{(i_1)} \wedge (e \cdot f_{(i_2)}) \wedge \ldots \ + \ldots \\ &+ f_{(i_1)} \wedge \ldots \wedge (e \cdot f_{(m)}) \wedge f_{(m+1)} \ldots \ + \ \ldots \ . \end{aligned} \tag{78}$$

The $\wedge$ indicates how to calculate the result. The rules are (Φ, Ψ and v are neighbour pieces)

$$\Phi \wedge f_j \wedge \Psi \wedge f_i \wedge v := -\Phi \wedge f_i \wedge \Psi \wedge f_j \wedge v, \quad j > i \tag{79}$$

$$\Phi \wedge f_i \wedge \Psi \wedge f_i \wedge v := 0 \tag{80}$$

$$\Phi \wedge \left(\sum_{i=1}^{r} c_i f_i \right) \wedge v := \sum_{i=1}^{r} c_i \left(\Phi \wedge f_i \wedge v \right) . \tag{81}$$

The definition (78) makes sense if there are only finitely many terms on the right hand side. A closer examination [4] shows that this only works for subalgebras of KN. We introduce the following subalgebras

$$\mathrm{KN}_+ := \langle\, e_n \mid n \geq 2 \,\rangle \tag{82}$$

$$\mathrm{KN}_-^l := \langle\, e_n^l \mid n \leq -3g \,\rangle \quad l = 1, \ldots, k-1 \tag{83}$$

For the elements of KN_+ and of KN_-^l the action (78) is well defined. Hence H is a module over these algebras. Unfortunately this is not true for KN_-^l, $l \geq 2$.

In the case of KN_2 H becomes a module over a central extension of KN [**1**]. It is reasonable to expect that this will be true in the general situation. This question and a closer study of the central extensions of KN_k is under investigation [**5**][**4**].

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