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Bayes and Credibility Premiums

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Abstract. For a risk whose annual claim amounts are conditionally i.i.d. with respect to a risk parameter, it is known that the Bayes and the credibility premiums are asymptotically optimal in terms of losses. In the present note it is shown that the Bayes and the credibility premiums actually converge to the individual premium.

Keywords. Individual premium, Bayes premium, credibility premium, asymptotic optimality, convergence.

1. Introduction.

In this note we consider the classical model in experience rating:

Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $L^2(\mathcal{F})$ denote the Hilbert of all random variables $\Omega \longrightarrow \mathbb{R}$ having finite second moments, and consider $\Theta, X_1, \ldots, X_n, X \in L^2(\mathcal{F})$. These random variables are interpreted as follows:

- Θ is a risk parameter which is observable or not and which determines the joint distribution of the annual claim amounts of the risk;
- $X_1, \ldots, X_n$ are observable annual claim amounts of the risk in n years in the past; and
- X is the annual claim amount of the risk in a future year which is to be predicted by a premium $\delta^* \in \Delta$ minimizing the loss $E[X-\delta]^2$ over Δ , where $\Delta \subseteq L^2(\mathcal{F})$ is a prescribed class of premiums to be specified below.

We assume that the following conditions are fulfilled:

- $X_1, \ldots, X_n, X$ are conditionally independent with respect to Θ ;
- $X_1, \ldots, X_n, X$ are conditionally identically distributed with respect to Θ ; and
- $0 < \text{var}\, E(X|\Theta)$.

Here $E(X|\Theta)$ denotes the conditional expectation of X with respect to the σ-algebra $\sigma(\Theta)$ generated by Θ , and we have $E(X|\Theta) \in L^2(\sigma(\Theta))$.

Define

$$\mu := EX = E[E(X|\Theta)] ,$$

$$\varphi := E[\text{var}(X|\Theta)] ,$$

$$\lambda := \text{var}\, E(X|\Theta) > 0 ,$$

$$\varkappa := \frac{\varphi}{\lambda} ,$$

and let $\overline{X}(n)$ denote the sample mean $\frac{1}{n} \sum_{i=1}^{n} X_i$.

We consider four classes of premiums:

$$\Delta_0 := \mathbb{R}$$
$$\bar{\Delta}_n := \text{lin}\,\{1, X_1, \ldots, X_n\}$$
$$\Delta_n := L^2(\sigma(X_1, \ldots, X_n))$$
$$\Delta_\infty := L^2(\sigma(\Theta)) \quad .$$

Since each of $\Delta_0, \bar{\Delta}_n, \Delta_n, \Delta_\infty$ is a closed subspace of $L^2(F)$, the projection theorem in Hilbert spaces yields the existence of unique $\delta_0^* \in \Delta_0$, $\bar{\delta}_n^* \in \bar{\Delta}_n$, $\delta_n^* \in \Delta_n$, $\delta_\infty^* \in \Delta_\infty$ satisfying

$$E[X-\delta_0^*]^2 = \inf_{\Delta_0} E[X-\delta]^2 \quad ,$$
$$E[X-\bar{\delta}_n^*]^2 = \inf_{\bar{\Delta}_n} E[X-\delta]^2 \quad ,$$
$$E[X-\delta_n^*]^2 = \inf_{\Delta_n} E[X-\delta]^2 \quad ,$$
$$E[X-\delta_\infty^*]^2 = \inf_{\Delta_\infty} E[X-\delta]^2 \quad ;$$

for a proof of the projection theorem, see e.g. Brockwell and Davis (1987).

In Section 2 of this note we recall some known results concerning the identification of

- the <u>collective premium</u> δ_0^* ,
- the <u>credibility premium</u> $\bar{\delta}_n^*$,
- the <u>Bayes premium</u> δ_n^* , and
- the <u>individual premium</u> δ_∞^*

and of the losses attached to these premiums. In Section 3 we study the asymptotic properties of the Bayes and the credibility premiums.

2. Known results.

Since $X, X_1, \ldots, X_n$ are conditionally independent with respect to Θ, the same is true for X and each $\delta \in \Delta_n$. This yields the following useful result:

2.1. Lemma.

$$E[X-\delta]^2 = E[X-E(X|\Theta)]^2 + E[E(X|\Theta)-\delta]^2$$

holds for all $\delta \in \Delta_n$.

For the optimum premiums δ_0^*, $\bar{\delta}_n^*$, δ_n^*, δ_∞^* we have:

2.2. Proposition.

(a) $\delta_0^* = \mu$.

(b) $\bar{\delta}_n^* = \frac{\varkappa}{\varkappa+n} + \frac{n}{\varkappa+n}\bar{X}(n)$.

(c) $\delta_n^* = E(\,E(X|\Theta)\;|X_1,\ldots,X_n) = E(\,X\;|X_1,\ldots,X_n)$.

(d) $\delta_\infty^* = E(X|\Theta)$.

For the losses attached to these premiums we have:

2.3. Proposition.

(a) $E[X-\delta_0^*]^2 = \varphi + \lambda$.

(b) $E[X-\bar{\delta}_n^*]^2 = \varphi + \frac{\varkappa}{\varkappa+n}\lambda$.

(c) $E[X-\delta_n^*]^2 = \varphi + E[\mathrm{var}(\,E(X|\Theta)\;|X_1,\ldots,X_n)]$.

(d) $E[X-\delta_\infty^*]^2 = \varphi$.

In particular,

$$E[X-\delta_\infty^*]^2 \leq E[X-\delta_n^*]^2 \leq E[X-\bar{\delta}_n^*]^2 \leq E[X-\delta_0^*]^2 .$$

In Propositions 2.2 and 2.3, assertions (a) and (d) are immediate, and assertion (c) is an easy consequence of Lemma 2.1. Assertion (b) of Proposition 2.2 is due to Bühlmann (1967,1970); it follows from the fact that $\bar{\delta}_n^*$ being the projection of X onto $\text{lin}\,\{1,X_1,\ldots,X_n\}$ satisfies $E[(X-\bar{\delta}_n^*)Z] = 0$ for all $Z \in \{1,X_1,\ldots,X_n\}$, and hence $EX = E\bar{\delta}_n^*$ and $\text{cov}(X,X_j) = \text{cov}(\bar{\delta}_n^*,X_j)$ for all $j \in \{1,\ldots,n\}$. Assertion (b) of Proposition 2.3 is due to Jewell (1976) and is obtained by computing $\text{var}(\, X - \frac{n}{\varkappa+n}\,\bar{X}(n)\,)$.

3. Asymptotic considerations.

By Proposition 2.3, the individual premium δ_∞^* is optimal, and both the Bayes premium δ_n^* and the credibility premium $\bar{\delta}_n^*$ are asymptotically optimal in the sense that they satisfy

$$E[X-\delta_\infty^*]^2 = \lim E[X-\delta_n^*]^2 = \lim E[X-\bar{\delta}_n^*]^2 .$$

We shall now prove a stronger result, showing that the Bayes and the credibility premiums converge to the individual premium:

3.1. Theorem.

(a) $$\lim E[\delta_\infty^*-\delta_n^*]^2 = \lim E[\delta_\infty^*-\bar{\delta}_n^*]^2 = 0 .$$

(b) $$\lim |\delta_\infty^*-\delta_n^*| = 0 \quad \text{a.s.}(P) .$$

Proof. By Lemma 2.1 and Propositions 2.2 and 2.3 we have

$$\begin{aligned} E[\delta_\infty^*-\bar{\delta}_n^*]^2 &= E[X-\bar{\delta}_n^*]^2 - E[X-\delta_\infty^*]^2 \\ &= (\varphi + \frac{\varkappa}{\varkappa+n}\lambda) - \varphi \\ &= \frac{\varkappa}{\varkappa+n}\lambda \end{aligned}$$

and, similarly,

$$E[\delta_\infty^*-\delta_n^*]^2 \le \frac{\varkappa}{\varkappa+n}\lambda ,$$

which proves (a).

By Proposition 2.2, we have

$$\delta_n^* = E(\, E(X|\Theta) \,|X_1,\ldots,X_n) ,$$

which means that the sequence $\{\delta_n^*\}$ is a martingale. Since $E(X|\Theta) \in L^2(F)$, there exists a unique $Z \in L^2(F)$ satisfying

$$\lim E[Z-\delta_n^*]^2 = 0$$

and

$$\lim |Z-\delta_n^*| = 0 \quad \text{a.s.}(P) ;$$

see e.g. Neveu (1972). In view of (a), we have $Z = \delta_\infty^*$ a.s.(P) , which proves (b). □

With regard to the martingale property of the sequence $\{ \delta_n^* \}$ of Bayes premiums, it is interesting to note that the sequence $\{ \bar{\delta}_n^* \}$ of credibility premiums usually fails to be a martingale. In fact, we have

$$E(\bar{\delta}_{n+1}^* | X_1, \ldots, X_n) = \frac{\varkappa+n}{\varkappa+n+1} \bar{\delta}_n^* + \frac{1}{\varkappa+n+1} \delta_n^* ,$$

which shows that $\{ \bar{\delta}_n^* \}$ is a martingale if and only if the Bayes and the credibility premiums agree for each sample size n .

References

Brockwell, P.J., and Davis, R.A. (1987) Time Series: Theory and Methods. Springer, Berlin - Heidelberg - New York.

Bühlmann, H. (1967) Experience rating and credibility. ASTIN Bulletin 4, 199-207.

Bühlmann, H. (1970) Mathematical Methods in Risk Theory. Springer, Berlin - Heidelberg - New York.

Jewell, W.S. (1976) A survey of credibility theory. Operations Research Certer Research Report No. 76-31. University of California, Department of Industrial Engineering and Operations Research, Berkeley.

Neveu, J. (1972) Martingales à Temps Discret. Masson, Paris.

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