SonderForschungsBereich 504

Rationalitätskonzepte,
Entscheidungsverhalten und
ökonomische Modellierung

No. 08-31

Neo-additive capacities and updating

Jürgen Eichberger*
and Simon Grant**
and Jean-Philippe Lefort***

December 2008

Financial support from the Deutsche Forschungsgemeinschaft, SFB 504, at the University of Mannheim, is gratefully acknowledged.

*Sonderforschungsbereich 504, email: juergen.eichberger@awi.uni-heidelberg.de

**Department of Economics Rice University, email: sgrant@rice.edu

***Sonderforschungsbereich 504, email: jean-philippe.lefort@awi.uni-heidelberg.de

Universität Mannheim
L 13,15
68131 Mannheim

Neo-additive capacities and updating

Jürgen Eichberger, Simon Grant Jean-Philippe Lefort

University of Heidelberg Rice University University of Heidelberg

This version: December 17, 2008

Abstract

This note shows that capacities satisfying the axioms *consquentialism, state independence* and *conditional certainty equivalent consistency* under updating are a generalised version of neo-additive capacities as axiomatised in Chateauneuf, Eichberger, and Grant (2007).

1 Introduction

A major challenge for modelling ambiguity of a decision maker in a dynamic context lies in the well-known precarious relationship between updating capacities or multiple priors, dynamic consistency and consequentialism. Early work by Epstein and LeBreton (1993) and Eichberger and Kelsey (1996) showed that updating Choquet Expected Utility (CEU) preferences, which satisfy consequentialism, in a dynamically consistent way implied additive beliefs. Even if dynamic consistency was constrained to an event tree, ambiguous beliefs modelled by a capacity were possible only on the final partition of events (see for example, Sarin and Wakker (1998) and Eichberger, Grant, and Kelsey (2005a)). For ambiguity beliefs modelled with multiple priors, Epstein and Schneider (2003) found that the set of priors had to fulfill a fairly restrictive rectangularity condition in order to guarantee dynamically consistent preferences. In particular, the original Ellsberg paradox cannot be explained with rectangular sets of priors.

In the light of these results, there are essentially two ways to proceed. Either we can abandon consequentialism, and all the models that rely on it such as CEU and multiple priors, or we can relax dynamic consistency. The former route has been explored by Hanany and Kilbanoff (2004).

In the spirit of Gilboa and Schmeidler (1993), we consider a preference relation and the family of its updated preferences which satisfy the three axioms *Consequentialism*, *State Independence,* and *Conditional Certainty Equivalent Consistency*. For the case where the beliefs can be described by multiple priors, as in the approach of Gilboa and Schmeidler (1989), Pires (2002) proved that these three axioms are equivalent to the *Full Bayesian Updating* of all prior probabilities. For the case where the preference relation can be represented by a Choquet integral and beliefs by a capacity, as in Schmeidler (1989), Eichberger, Grant, and Kelsey (2005b) and Horie (2007) established that *Consequentialism*, *State Independence,* and *Conditional Certainty Equivalent Consistency for Binary Gambles*, are equivalent to *Full Bayesian Updating* of the capacity as suggested by Jaffray (1992) and Walley (1991).[1]

In this note, we characterize the family of capacities for which the three initial axioms of Pires (2002) hold, that is, where *Conditional Certainty Equivalent Consistency* is not restricted to binary acts. Interestingly, it turns out to be a class of capacities which is slightly more general than *neo-additive capacities* which were axiomatized in Chateauneuf, Eichberger, and Grant (2007). For neo-additive capacities the Choquet expected utility preferences can be calculated as a convex combination of expected utility with respect to an additive probability distribution and the *Hurwicz criterion* (Hurwicz (1951)) which itself is a convex combination of the utility values of the best and the worst outcomes.

The Choquet integral with respect to a *generalized neo-additive capacity* is a linear combination of the expected utility and the Hurwicz criterion, but the combination need not be convex. Moreover, we can show that *convex generalized neo-additive capacities* are the only capacities for which the core of the Full Bayesian update of a capacity coincides with the set of Bayesian updates of the probabilities in the core of the original capacity. These results provides a further justification for *neo-additive capacities* as a useful restriction on the Choquet expected utility approach.

2 The model

Let Ω be a finite set of states of the world, $\Sigma = 2^{\Omega}$, the set of events in Ω. For $E \in \Sigma$, E^c denotes the complement of E. Let X be a set of outcomes. An *act* is a function $f : \Omega \to X$, and $\mathcal{F}$ denotes the set of such acts. A *capacity* v is a set function from Σ to $\mathbb{R}$ with $v(\emptyset) = 0$, $v(\Omega) = 1$ and $v(A) \le v(B)$ for all $A \subset B$, A and B in Σ.

In the main part of the paper, we will restrict attention to capacities for which the only null set is the empty set, i.e., $v(E) = 0 \Leftrightarrow E = \emptyset$, and the only universal set is Ω, $v(E) = 1 \Leftrightarrow E = \Omega$. The general case is treated in Appendix B.

Given a *von Neumann-Morgenstern utility function* $u : X \to \mathbb{R}$, the *Choquet Expected Utility*

[1] Horie (2007) showed that the necessary conditions in Eichberger, Grant, and Kelsey (2005b) were too stong and suggested an appropriate weakening of the Conditional Certainty Equivalence Consistency Axiom.

(CEU) of an act f with respect to the capacity ν is given by $CEU(f,\nu) = \int_{-\infty}^{0}(v(u(f(\omega)) \geq t) - 1)dt + \int_{0}^{+\infty} v(u(f(\omega)) \geq t)dt$. Since acts are finite-valued they can be written as $f = \sum_{i=1}^{n} x_i A_i$, where A_i is the indicator function of the set A_i. Without loss of generality, suppose that the finite outcomes $x_i \in f(\Omega)$ are ordered such that $u(x_i) < u(x_{i+1})$,[Simon asks (16/12) shouldn't "$<$" be "$\leq$"] then

$$\begin{aligned} CEU(f,\nu) &= \sum_{i=1}^{n} u(x_i) \cdot [v(A_i \cup A_{i+1} \cup ... \cup A_n) - v(A_{i+1} \cup A_{i+2} \cup ... \cup A_n)] \\ &= \sum_{i=1}^{n} u(x_i) \cdot m(A_i), \end{aligned}$$

with $m(A_i) := [v(A_i \cup A_{i+1}..A_n) - v(A_{i+1} \cup A_{i+2}..A_n)]$.

Throughout this paper, we will consider preference relations $\succsim$ on $\mathcal{F}$ which can be represented by a *Choquet Expected Utility (CEU)* functional,

$$f \succsim g \quad \text{if and only if} \quad CEU(f,\nu) \geq CEU(g,\nu).$$

2.1 Updating preferences

Consider a family of preference relations $\succsim_E$ on $\mathcal{F}$ which represent the decision maker's preferences after it becomes known that the event E has occurred. The ex-ante unconditional preference relation on $\mathcal{F}$ will be denoted by $\succsim$.

For preferences which are additive, that is, represented by a CEU functional with an additive capacity ν, standard Bayesian updating satisfies the following two axioms.

Axiom C *Consequentialism*

For any two acts $f, g \in \mathcal{F}$,

$$\text{if } f = g \text{ on } E, \quad \text{then} \quad f \sim_E g.$$

Consequentialism rules out effects on future choices from outcomes which would have become relevant in the event E^c which did not happen.

In contrast, the second axiom , dynamic consistency, requires that preferences after E occurred remain consistent with ex-ante preferences.

Axiom DC *Dynamic Consistency*

For any acts $f, g \in \mathcal{F}$ and any event $E \in \Sigma$,

$$f \succ_E g \quad \text{if and only if} \quad f_E g \succ g.$$

It is well known (Ghirardato (2002)), that the two axioms, consequentialism and dynamic consistency, imply a capacity ν for a CEU decision maker which is additive and updated by Bayes rule . Hence, any updating rule which leaves room for uncertainty represented by a non-additive capacity must relax either consequentialism or dynamic consistency.

Retaining consequentialism Pires (2002) proposes a weaker version of DC, *conditional certainty equivalent consistency* which restricts the act g of the classical DC axiom to be constant.

Axiom CCEC *(Conditional Certainty Equivalent Consistency)*

For any event $E \neq \varnothing$, any outcome $x \in X$, and any act f in $\mathcal{F}$,

$$f \sim_E x \quad \text{if and only if} \quad f_E x \sim x.$$

For multiple-prior preferences, Pires (2002) proved that *consequentialism* and *conditional certainty equivalent consistency* imply the Full Bayesian updating rule, where each probability distribution in the set of priors is updated according to Bayes rule. .

3 Generalized neo-additive capacities

In this section we introduce a slightly generalized concept of a neo-additive capacity, which we call *Generalized Neo-additive Capacity (GNAC)*. CEU preferences with a neo-additive capacity, as axiomatized in Chateauneuf, Eichberger, and Grant (2007), are a special case of a GNAC. As a first building block we introduce a capacity which we call *Hurwicz capacity,* because it was implicit in the decision rule suggested by Hurwicz (1951).

Definition 3.1 *A Hurwicz capacity μ_α with degree of optimism α is defined by $\mu_\alpha(\emptyset) = 0$, $\mu_\alpha(\Omega) = 1$, and $\mu_\alpha(E) = \alpha$ for all other events $E \in \Sigma$.*

[Simon asks (16/12) for μ_α to be a capacity, don't we require $\alpha \in 0, 1$?]

A *generalized neo-additive capacity (GNAC)* can now be defined.

Definition 3.2 *For a finitely additive measure probability measure π on (Ω, Σ) and a pair of numbers (δ, α), with $\delta \leq 1$, a* generalized neo-additive capacity v *is defined as*

$$v(E|\pi, \delta, \alpha) = \delta \cdot \mu_\alpha(E) + (1 - \delta) \cdot \pi(E)$$

for all $E \in \Sigma$.

A *neo-additive capacity* is the special case of a GNAC which satisfies the additional constraints $\delta \geq 0$ and $0 \leq \alpha \leq 1$.

The Choquet expected value of an act f with respect to the GNAC $v(E|\pi, \delta, \alpha)$ is given by:

$$CEU(f) = (1-\delta)E_\pi(u \circ f) + \delta(\alpha \cdot \max\{x : x \in u \circ f(\Omega)\} + (1-\alpha) \cdot \min\{x : x \in u \circ f(\Omega)\})$$

We will prove now that the only CEU preferences satisfying the Axiom CCEC are generalized neo-additive capacities. Hence, for decision makers with CEU preferences which update their beliefs according to the Full Bayesian updating rule, the stronger Axiom CCEC implies that capacities must be GNAC. Indeed, for CEU preferences with Full Bayesian updating, Axiom CCE almost characterizes neo-additive capacities.

Proposition 3.1 *CEU preferences satisfy Axiom CCEC if and only if the capacity v is a GNAC.*

The following remark indicates the way in which a small generalization of this result is possible.

Remark 3.1 *Two remarks are in order.*

(i) It is worth noting that our proof uses only one way of Axiom CCEC, namely $f \sim_E x \Rightarrow f_E x \sim x$.

(ii) In the statement of Axiom CCEC, we can replace the constant act x by slightly more general acts:

Alternative Axiom*: Suppose* $\arg\min_{\omega \in \Omega} f(\omega) \cap \arg\min_{\omega \in \Omega} g(\omega) \neq \varnothing \subset A$ *and* $\arg\max_{\omega \in \Omega} f(\omega) \cap$

$\arg\max_{\omega\in\Omega} g(\omega) \neq \varnothing \subset A$*, then for any* $h \in \mathcal{F}$ *such that*

$$\max\left\{\min_{\omega\in\Omega} f(\omega), \min_{\omega\in\Omega} g(\omega)\right\} \leq \min_{\omega\in\Omega} h(\omega), \quad \max_{\omega\in\Omega^c} h(\omega) \leq \min\left\{\max_{\omega\in\Omega} f(\omega), \max_{\omega\in\Omega} g(\omega)\right\}.$$

$$f \sim_A g \quad \textit{if and only if} \quad f_A h \sim g_A h.$$

This alternative axiom is stronger than CCEC but, for CEU preferences, it is equivalent to CCEC. Hence, for CEU preferences, Axiom CCEC implies GNAC which in turn implies the alternative axiom.

Let us check that GNAC satisfy this axiom: let $\arg\min_{\omega\in\Omega} f(\omega) \cap \arg\min_{\omega\in\Omega} g(\omega) = E_m$ *and* $\max_{\omega\in\Omega} f(\omega) \cap \arg\max_{\omega\in\Omega} g(\omega) = E_M$*. Let* $p = (1-\delta)\pi + \alpha\delta d_{E_m} + (1-\alpha)\delta d_{E_M}$*, where* d_E *denotes the Dirac measure of the set* E*. As* $\max\left\{\min_{\omega\in\Omega} f(\omega), \min_{\omega\in\Omega} g(\omega)\right\} \leq \min_{\omega\in\Omega} h(\omega)$, $\max_{\omega\in\Omega^c} h(\omega) \leq \min\left\{\max_{\omega\in\Omega} f(\omega), \max_{\omega\in\Omega} g(\omega)\right\}$ *then* $\int f_A h dv = \int f_A h dp$ *and* $\int g_A h dv = \int g_A h dp$*. As* E_m *and* E_M *are included in* A*, then* $\int f dv_A = \int f dp_A$ *and* $\int g dv_A = \int g dp_A$*. Therefore CCEC comes from that the same measure is used at every stage.*

4 Convex GNAC

Axiom CCEC is satisfied for *Multiple Priors (MP)* preferences if all prior probability distributions are updated according to the Bayesian rule (Pires (2002)). It is well known that CEU and MP preferences coincide if and only if the capacity of a Choquet expected-utility maximizer is convex. As Horie (2007) has shown, however, Axiom CCEC is no longer satisfied for CEU with Full Bayesian updating. In order to see why, we need some new notation.

Let v be a convex capacity and $C(v) = \{m \in \Delta(\Omega) | \ m \geq v\}$ its core. Furthermore, let $P_E = \{\frac{p}{p(E)} | \ p \in C(v)\}$ be the set of Bayesian updates of the probabilities in the core $C(v)$. As Horie (2007) points out, $P_E \subseteq C(v_E)$. This follows since $p \in C(v_E)$ and $A \subset E$ imply

$$\begin{aligned}\frac{p(A)}{p(E)} - v_E(A) &= \frac{p(A)}{p(E)} - \frac{v(A)}{v(A) + \overline{v}(A^c \cap E)} \\ &= \frac{p(A)(v(A)+\overline{v}(E\backslash A)) - v(A)p(E)}{p(E)(v(A) + \overline{v}(E\backslash A))} \\ &= \frac{p(A)\overline{v}(E\backslash A) - v(A)p(E\backslash A)}{p(E)(v(A) + \overline{v}(E\backslash A))}.\end{aligned}$$

As $p \in C(v)$, we have $p(A) \geq v(A)$ and $p(E\backslash A) \leq \overline{v}(E\backslash A)$. Hence, $\frac{p(A)}{p(E)} \leq v_E(A)$ and

$P_E \subseteq C(v_E)$. Horie (2007) shows by example that $P_E \subsetneq C(v_E)$.

For a convex capacity with $P_E = C(v_E)$ for all $E \in \Sigma$, we can apply Pires' result in order to see that CCEC holds. Our next proposition shows that $C(v_E) \subseteq P_E$ if and only if v is a convex GNAC, when $|\Omega| > 3$.

Proposition 4.1 *If $|\Omega| > 3$, $C(v_E) = P_E$ if and only if v is a* convex *GNAC.*

We conclude this section with a couple of remarks.

Remark 4.1 *The following remarks are in order:*
(i)Convex neo-additive capacities are ϵ-contaminations. If the state space Ω is finite, then there exist convex GNAC which are not ϵ-contaminations. For example, $n = 4$ and $\pi(E) = \frac{|E|}{|\Omega|}$, then $v = \frac{6}{5}\pi - \frac{1}{5}$ is convex, but not an $\varepsilon-$contamination. With a non-atomic state space Ω, however, monotonicity implies that the only GNAC are ϵ-contaminations. With neo- additive capacities, the only way to be pessimistic consists in overweighting the minimum of an act and to undervalue all other outcomes. For a GNAC, however, there is the possibility of underweighting the maximum and overweighting all other outcomes.
(ii)Proposition 4.1 provides a necessary and sufficient condition for capacities to guarantee $C(v_E) = P_E$. An alternative condition can be found in Theorem 2 of Jaffray (1992). Theorem 4.1, however, holds for convex capacities whereas Jaffray's Theorem 2 is true only for belief functions.
(iii) If $|\Omega| = 3$ holds, then $C(v_E) = P_E$ is true for every convex capacity.
(iv) Note that it follows from Proposition 3.1 that the only case in which $C(v_E) = P_E$ holds for convex capacities is when Axiom CCEC is true.

5 Conclusion

In this note, we show that a decision maker with CEU preferences satisfying *Consequentialism*, *State Independence,* and *Conditional Certainty Equivalent Consistency* will hold beliefs which are almost neo-additive. Such preferences can be represented by as a linear combination of the expected utility with respect to some additive probability distribution and the maximum and minimum utility over outcomes. Moreover, if beliefs are represented by a convex capacity then the core of the Full Bayesian updated capacities equals the set of Bayesian updates of the probabilities in the core of the prior capacity. These observations clarify some open questions on Fully Bayesian updating of capacities and multiple priors and provide additional arguments for considering neo-additive capacities in a dynamic context.

Appendix A. Proofs

Proof of Proposition 3.1:

Step 1: Let $f = \sum_{j=1}^{n} x_j A_j$ where A_j is the indicator function of the set A_j and $u(x_j) < u(x_{j+1})$, then $\int f dv = \int f dm$ with m measure such that for all j $v(A_j ... A_n) = m(A_j ... A_n)$.

Lemma A.1 *v satisfies CCEC then for any A_i with $i \neq 1, n$ we have $\int f dv_{A_i^c} = \frac{1}{m(A_i^c)} \int f dm_{A_i^c}$, where $\int f dv_{A_i^c}$ is the Choquet integral of f when A_i^c has occurred and $\int f dm_{A_i^c}$ is integral of f calculated according to m updated by Bayes rule.*

That lemma means that, for those A_i with $i \neq 1, n,$ i.e.. the ones for which the values are not the extreme ones, the Choquet integral of f is calculated according to the same measures wether it is updated or not. Namely we have $\int f dv = \int f dm,$ let $\int f dv_{A_i^c} = \int f dp,$ we are going to prove $\frac{m}{m(A_i^c)} = p.$

Proof. Let y be the certainty equivalent of f conditional on $A_i^c,$ $f \sim_{A_i^c} y$. If $u(y) \leq u(x_{i-1}),$ we define g on A_i^c as $g = z$ on A_n and $g = f$ else. By choosing $u(z) > u(x_i)$ we get g such that $g \sim_{A_i^c} x$ with $u(x_{i-1}) < u(x) < u(x_{i+1})$. By continuity, this is possible. If $u(x_{i+1}) \leq u(y)$ we define another g, such that $g \sim_{A_i^c} x$ with $u(x_{i-1}) < u(x) < u(x_{i+1}),$ by decreasing x_1. f and g are comonotonic because g is different of f only on the lowest value of f becoming lower or the highest value becoming higher, therefore $\int g dv_{A_i^c} = \int g dp$. Now, we make use of CCEC and get $g_{A_i^c} x \sim x$. As $u(x_{i-1}) < u(x) < u(x_{i+1}),$ f and $g_{A_i^c} x$ are comonotonic so their Choquet integrals are computed according to the same measure, namely $\int g_{A_i^c} x dv = \int g_{A_i^c} x dm$. Now we get $\int g_{A_i^c} x dv = u(x). \int_{A_i^c} gdm + m(A_i)u(x) = u(x)$ so

$$u(x) = \frac{1}{m(A_i^c)} \int_{A_i^c} gdm = \int_{A_i^c} gdv_{A_i^c} = \int_{A_i^c} gdp.$$

Let $\pi = \frac{m}{m(A_i^c)} - p$, we have $\int_{A_i^c} gd\pi = 0$. Let us prove $\pi = 0$. For each A_j, $j \neq i$ we define g_j as follows: $g_j = g$ on A_j^c and $g_j = x_{\epsilon j}$ on A_j^c. We have $g_j \sim_{A_i^c} x_j'$. By continuity of u, we can select $x_{\epsilon j}$ close to x_j and get $u(x_{i-1}) < u(x_j') < u(x_{i+1})$ {rem here $j \neq i \pm 1$}. g_j and g are comonotonic therefore $g_{jA_i^c} x$ is calculated with the same measure m and also g_j updated when A_i has occurred with the same p. So with the same reasoning as above we get $\int_{A_i^c} g_j d\pi = 0$. The g_j are independent vectors so $\pi = 0$. Therefore, we obtain the result announced. ■

Step 2. We have $\int f dv_{A_i^c} = \frac{1}{m(A_i^c)} \int_{A_i^c} fdm$. That is true for each measure m such that $\int fdv = \int fdm$ whatever the ranking of A_i provided it is not extreme. So for two such measures m and m' (let us say that for m, $u(x_i)$ is between $u(x_j)$ and $u(x_{j+1})$ and for m', $u(x_i)$ is between $u(x_{j'})$ and $u(x_{j'+1})$), we have:

$$\frac{1}{m(A_i^c)} \int_{A_i^c} fdm = \frac{1}{m'(A_i^c)} \int_{A_i^c} fdm'$$

that equality is true for all g such that $g = \sum_{j=1, j\neq i}^{n} x_j A_j$ and g comonotonic with f. So $m(A_i^c) = m'(A_i^c)$. We have $m(A_i^c) = 1 - m(A_i) = 1 - v(A_i \cup A_{j+1}...A_n) + v(A_{j+1}...A_n)$, and $m'(A_i^c) = 1 - m'(A_i) = 1 - v(A_i \cup A_{j'+1}...A_n) + v(A_{j'+1}...A_n)$

$$v(A_i \cup A_{j+1}...A_n) - v(A_{j+1}...A_n) = v(A_i \cup A_{j'+1}...A_n) - v(A_{j'+1}...A_n)$$

This is true for any f, let $A_i = E$ as A_i $i \neq 1, n$, let $F = A_{j+1}...A_n$ let $G = A_{j'+1}...A_n$ the first hand of the equality holds if $v(A_i \cup A_{j+1}...A_n) - v(A_{j+1}...A_n) \neq 1$, i.e. $v(F) \neq 0$ and $v(F \cup E) \neq 1$ (which insures us that $v_{A_i^c}$ exists), the second hand of the equality is available for every G such that $v(G) \neq 0$ and $v(G \cup E) \neq 1$ (which insures us that $m(A_i^c) \neq 0$). So we get:

$$v(F \cup E) - v(F) = v(G \cup E) - v(G)$$

Step 3. Now we make use of Proposition 3.1 of Chateauneuf, Eichberger, and Grant (2007). This proposition states four properties which are equivalent to being a neo-additive capacity.

From the proof of that proposition, it turns out that, without other null sets than the empty set, a GNAC is a capacity which fulfills condition (a). So we can conclude that the capacity v is a GNAC. ■

Proof of Proposition 4.1:

Let us suppose $C(v_E) = P_E$. $C(v_E)$ is the core of a convex capacity. It is known, see e.g. Delbaen (1974), that for any maximal chain (a chain is an ordered set of sets) $C_1 \subset ..C_i.. \subset E$ we have got $\mu \in C(v_E)$ such that $\forall i\ \mu(C_i) = v_E(C_i)$. $\mu \in P_E$ so for all i there exists $p \in C(v)$ such that,

$$\frac{p(C_i)}{p(E)} = v_E(C_i) = \frac{v(C_i)}{v(C_i) + \overline{v}(E \backslash C_i)}$$

Which means from computations made above that $p(C_i)\overline{v}(E\backslash C_i) - v(C_i)p(E\backslash C_i) = 0$. As $p(C_i) \geq v(C_i)$ and $p(E\backslash C_i) \leq \overline{v}(E\backslash C_i)$, we get $p(C_i) = v(C_i)$ and $p(E\backslash C_i) = \overline{v}(E\backslash C_i)$. From (1) we deduce that for all i,

$$p(E) = v(C_i) + \overline{v}(E\backslash C_i) = 1 + v(C_i) - v(C_i \cup E^c)$$

so for A and B non void strictly included in E and ordered by inclusion we have,

$$v(E^c \cup A) - v(A) = v(E^c \cup B) - v(B)$$

We can prove it remains true if A and B are not ordered by inclusion. if $A \cap B \neq \emptyset$, we have,

$$v(E^c \cup A) - v(A) = v(E^c \cup (A \cap B)) - v(A \cap B) = v(E^c \cup B) - v(B)$$

if $A \cup B \subsetneq E$ we do the same with $A \cup B$:

$$v(E^c \cup A) - v(A) = v(E^c \cup (A \cup B)) - v(A \cup B) = v(E^c \cup B) - v(B)$$

The remaining case is $A \cup B = E$ and $A \cap B = \emptyset$, if $|E| > 2$, we pick a non void set included in A or B, say A', and get,

$$v(E^c \cup A) - v(A) = v(E^c \cup A') - v(A') = v(E^c \cup (A' \cup B)) - v(A' \cup B) = v(E^c \cup B) - v(B)$$

if $|E| = 2$, as $|\Omega| > 3$ we can write $E^c = F \cup G$ and get,

$$(i): \quad v(F \cup G \cup A)) - v(G \cup A) = v(F \cup G \cup B) - v(G \cup B)$$

$$(ii): \quad v(G \cup A) - v(A) = v(G \cup B) - v(B)$$

$$(i) - (ii): \quad v(E^c \cup A)) - v(A) = v(E^c \cup B) - v(B)$$

So we get the property (a) which insures that v is a GNAC.

Conversely, let us suppose that v is a GNAC, we just need to prove that any extreme point μ of $C(v_E)$ belongs to P_E. There exits a maximal chain $C_1 \subset ..C_i..C_k \subset E$ such that $\forall i$, $\mu(C_i) = v_E(C_i)$. We are going to construct $p \in C(v)$ such that for all i,

$$\frac{p(C_i)}{p(E)} = v_E(C_i) = \frac{v(C_i)}{v(C_i) + \overline{v}(E \backslash C_i)}$$

On $\mathcal{P}(E)$, the set of parts of E, $v_{/E}$, v restricted to E is a convex capacity, so we can find in its core a probability p such that $p(C_i) = v(C_i)$ and $p(E) = v(C_k) + \overline{v}(E \backslash C_k)$, (compare Delbaen (1974)). By the Hahn Banach Theorem, we can extend p to Σ with p in the core of v. As v satisfies property (a) we have

$$p(E) = v(C_i) + \overline{v}(E \backslash C_i) = 1 + v(C_i) - v(C_i \cup E^c).$$

Hence,

$$p(C_i \cup E^c) = p(E) + p(C_i) = v(C_i \cup E^c).$$

Thus, p satisfies property (a) and we have $C(v_E) = P_E$. ■

Appendix B. Null sets

In this appendix we show that, with the appropriate modifications, one can extend the results of this paper to the case of general null and universal sets which was treated in Chateauneuf, Eichberger, and Grant (2007).

For any capacity v, there exists a partition of Ω into the set of null events $\mathcal{N}$, the set of universal events $\mathcal{U}$, the set of essential events $\mathcal{E}$. Those sets have the following properties: $\emptyset \in \mathcal{N}$; if $A \in \mathcal{N}$ then $B \in \mathcal{N}$, for every $B \subset A$; $A \in \mathcal{N}$ and $B \in \mathcal{N}$ then $A \cup B \in \mathcal{N}$, $\mathcal{U} = \{E \in \Sigma;$ $E^c \in \mathcal{N}\}$. The definitions of section 2 are modified on the following way:

Definition B.1 *(Chateauneuf, Eichberger, and Grant (2007)).The Hurwicz capacity exactly congruent with* $\mathcal{N}$ *and* α *degree of optimism is defined to be*

$\mu_\alpha^{\mathcal{N}} = 0$ if $E \in \mathcal{N}$; $\mu_\alpha^{\mathcal{N}} = 1$ if $E \in \Omega \backslash \mathcal{N}$ and $\mu_\alpha^{\mathcal{N}} = \alpha$ else.

Definition B.2 *For a given set of null events* $\mathcal{N}$ *, a finitely additive measure probability measure* π *on* (Ω, Σ) *that is null on* $\mathcal{N}$ *and a pair of number* (δ, α), *with* $\delta, \alpha \in [0, 1]$, *a neo-additive capacity* v *is defined as*

$$v(E/\mathcal{N}, \pi, \delta, \alpha) = \delta \cdot \mu_\alpha^{\mathcal{N}}(E) + (1 - \delta) \cdot \pi(E)$$

for all $E \in \Sigma$.

Definition B.3 *For a given set of null events* $\mathcal{N}$ *, a finitely additive measure probability measure* π *on* (Ω, Σ) *that is null on* $\mathcal{N}$ *and a pair of number* (δ, α), *with* $\delta \leq 1$, *a GNAC* v *is defined as*

$$v(E/\mathcal{N}, \pi, \delta, \alpha) = \delta \cdot \mu_\alpha^{\mathcal{N}}(E) + (1 - \delta) \cdot \pi(E)$$

for all $E \in \Sigma$.

The Choquet expected value of a simple function f with respect to the neo-additive capacity $v(E/\mathcal{N}, \pi, \delta, \alpha)$ is given by:

$$\int f dv = (1 - \delta) E_\pi(f) + \delta(\alpha \cdot \max\{x : f^{-1}(x) \notin \mathcal{N}\} + (1 - \alpha) \cdot \min\{y : f^{-1}(y) \notin \mathcal{N}\})$$

For general null sets, Proposition **3.1** remains valid.

Proposition B.1 *CEU preferences satisfy Axiom CCEC if and only if the capacity* v *is a GNAC.*

Proof. With null sets, a GNAC must fulfill not only condition (a), already mentioned, but also condition (d) of Proposition 3.1 of Chateauneuf, Eichberger, and Grant (2007).

A capacity v is a generalized neo-additive capacity if and only if

(a) for any three events $(E, F, G) \in \mathcal{E} \times \mathcal{E} \times \mathcal{E}$ such that $E \cap F = \emptyset = E \cap G$, $E \cup F \notin \mathcal{U}$ and

$E \cup G \notin \mathcal{U}$,

$$v(E \cup F) - v(F) = v(E \cup G) - v(G), \tag{B-1}$$

(d) for any $E \in \mathcal{E}$ and any $F \in \mathcal{N}$ such that $E \cap F = \emptyset$, $v(E \cup F) = v(E)$.

It remains to prove that (d) is also satisfied. Note first that Equation B-1 is satisfied for all sets E, F and G such that $(F) \neq 0$, $v(F \cup E) \neq 1$, $v(G) \neq 0$ and $v(G \cup E) \neq 1$. Let N be a null set and E an essential event we want to prove that $v(N \cup E) = v(E)$. By removing all the null sets of N^c, we get a set U such that $U \subset N^c$, $v(U) = 1$, $\forall A \subset U$, $v(A) = 0 \Leftrightarrow A = \emptyset$. We can apply Proposition 3.1 on U because there is no other null set than the void set on U and we get $v_{/U}$ is a GNAC, let us say $v_{/U}(E) = v(E/\pi, \delta, \alpha)$. There exists two essential sets A_2 and A_3 such that $A_2 \cup A_3 = U \setminus E$. Now we consider the algebra $\mathcal{A}$ whose atoms are $E \cup N = A_1$, A_2 and A_3, let us call v' the restriction of v to this algebra. On this algebra there is no other null set than the void one end do we can apply Proposition 3.1 so for $E \in \mathcal{A}$ we have $v'(E) = v(E/\pi', \delta', \alpha')$. Now, we have $v'(A_2) = \pi'(A_2) + \alpha'\delta'$, $v'(A_3) = \pi'(A_3) + \alpha'\delta'$, $v'(A_2 \cup A_3) = \pi'(A_2 \cup A_3) + \alpha'\delta'$ so $\alpha'\delta' = v'(A_2 \cup A_3) - v'(A_2) - v'(A_3)$. As the sets A_2 and A_3 belong to U, we have $\alpha\delta = v(A_2 \cup A_3) - v(A_2) - v(A_3)$ and, as for $E \in \mathcal{A}$, $v'(E) = v(E)$, we have $\alpha'\delta' = \alpha\delta$ and $\pi(A_2) = \pi'(A_2)$ and $\pi(A_3) = \pi'(A_3)$. As $v(U) = 1$ and $v(A_1 \cup A_2 \cup A_3) = 1$ we have $\pi(E) = 1 - \pi(A_2) - \pi(A_3)$ and $\pi'(A_1) = 1 - \pi'(A_2) - \pi'(A_3)$, therefore we have $v(E) = v(E \cup N)$. ■

References

CHATEAUNEUF, A., J. EICHBERGER, AND S. GRANT (2007): "Choice under Uncertainty with the Best and Worst in Mind: Neo-Addditive Capacities," *Journal of Economic Theory*, 137(1), 538 –567.

DELBAEN, F. (1974): "Convex Games and Extreme Points," *Journal of Mathematical Analysis and Applications*, 45, 210–233.

EICHBERGER, J., S. GRANT, AND D. KELSEY (2005a): "CEU Preferences and Dynamic Consistency," *Mathematical Social Sciences*, 49, 143–151.

EICHBERGER, J., S. GRANT, AND D. KELSEY (2005b): "Updating Choquet Expected Utility Preferences," *Mathematical Social Sciences*, 49, 143–151.

EICHBERGER, J., AND D. KELSEY (1996): "Uncertainty Aversion and Dynamic Consistency," *In-*

ternational Economic Review, 37, 625–640.

EPSTEIN, L. G., AND M. LEBRETON (1993): "Dynamically Consistent Beliefs Must Be Bayesian," *Journal of Economic Theory*, 61, 1–22.

EPSTEIN, L. G., AND M. SCHNEIDER (2003): "Recursive Multiple-Priors," *Journal of Economic Theory*, 113, 1–31.

GHIRARDATO, P. (2002): "Revisiting Savage in a Conditional World," *Economic Theory*, 20(1), 83–92.

GILBOA, I., AND D. SCHMEIDLER (1989): "Maxmin Expected Utility with a Non-Unique Prior," *Journal of Mathematical Economics*, 18, 141–153.

GILBOA, I., AND D. SCHMEIDLER (1993): "Updating Ambiguous Beliefs," *Journal of Economic Theory*, 59, 33–49.

HANANY, E., AND P. KILBANOFF (2004): "Updating Multiple Prior Preferences," *working paper, Northwestern University*.

HORIE, M. (2007): "Reexamination on Updating Choquet Beliefs," *Kyoto Institute of Economic Research Discussion Paper*, 148.

HURWICZ, L. (1951): "Optimiality Criteria for Decision Making under Ignorance," *Discussion paper 370, Cowles Comission*.

JAFFRAY, J.-Y. (1992): "Bayesian Updating and Belief Functions," *IEEE Transactions on Systems, Man and Cybernetics*, 22, 1144–1152.

PIRES, C. (2002): "A Rule for Updating Ambiguous Beliefs," *Theory and Decision*, 53, 137–152.

SARIN, R., AND P. WAKKER (1998): "Revealed Likelihood and Knightian Uncertainty," *Journal of Risk and Uncertainty*, 16, 223–250.

SCHMEIDLER, D. (1989): "Subjective Probability and Expected Utility without Additivity," *Econometrica*, 57, 571–587.

WALLEY, P. (1991): *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London.

Nr.	Author	Title
08-40	Florian Kutzner Tobias Vogel Peter Freytag Klaus Fiedler	Pseudocontingencies in stereotype formation: extending illusory correlations
08-39	Wendelin Schnedler Adam Dominiak	Uncertainty Aversion and Preference for Randomization
08-38	Susanne Abele Sandra I. Vaughan-Parsons Garold Stasser	Information Flow and Influence during Collective Search, Discussion, and Choice
08-37	Peter Freytag Tobias Vogel Florian Kutzner Klaus Fiedler	The reproduction of base-rates promotes pseudocontingencies
08-36	Klaus Fiedler Peter Freytag Thorsten Meiser	Pseudocontingencies: An Integrative Account of an Intriguing Cognitive Illusion
08-35	Alexander Unger Dagmar Stahlberg	The Influence of Self-Regulation on Decision Making
08-34	Dirk Simons Nicole Zein	Kosten aus einer asymmetrischen Informationsverteilung zwischen Abschlussprüfer und Mandant
08-33	Michael Bremert Axel Schulten	The Impact of Supervisory Board Characteristics on Firm Performance
08-32	Jürgen Eichberger Ani Guerdjikova	Case-Based Expected Utility: Preferences over Actions and Data
08-31	Jürgen Eichberger Simon Grant Jean-Philippe Lefort	Neo-additive capacities and updating
08-30	Vladimir I. Danilov Gleb A. Koshevoy Christine Lang	Equilibria with indivisible goods and package-utilities

Nr.	Author	Title
08-29	Thomas Gschwend	Electoral System Change in Belgium 2003: Party Strategies and Voter Responses
08-28	Michael Herrmann	Expectations about Coalitions and Strategic Voting under Proportional Representation
08-27	Karl-Martin Ehrhart Marion Ott Susanne Abele	Auction Fever: Theory and Experimental Evidence
08-26	Michel Regenwetter	Perspectives on Preference Aggregation
08-25	Daniel M. Bernstein Michael E. Rudd Edgar Erdfelder Ryan Godfrey Elizabeth F. Loftus	The Revelation Effect for Autobiographical Memory: A Mixture-Model Analysis
08-24	Markus Glaser Florencio Lopez de Silanes Zacharias Sautner	Looking Inside a Conglomerate: Efficiency of Internal Capital Allocation and Managerial Power Within a Firm
08-23	Markus Glaser Martin Weber	Financial Literacy und Anlegerverhalten
08-22	Markus Glaser Thomas Langer Jens Reynders Martin Weber	Scale Dependence of Overconfidence in Stock Market Volatility Forecasts
08-21	Patrick A. Müller Rainer Greifeneder Dagmar Stahlberg Kees Van den Bos Herbert Bless	The Role of Procedural Fairness in Trust and Trustful Behavior
08-20	Patrick A. Müller Rainer Greifeneder Dagmar Stahlberg Kees Van den Bos Herbert Bless	The Influence of Accessibility Experiences on the Willingness to Cooperate in Negotiations

Nr.	Author	Title
08-19	Jana Janßen Patrick A. Müller Rainer Greifeneder	A field study on the role of ease-of-retrieval in procedural justice judgments
08-18	Rainer Greifeneder Benjamin Scheibehenne	When choosing is difficult: Complexity and choice-overload
08-17	Paul Grout Wendelin Schnedler	Non-Profit Organizations in a Bureaucratic Environment
08-16	Clemens Kroneberg Isolde Heintze Guido Mehlkop	On shoplifting and tax fraud: An action-theoretic analysis of crime
08-15	Hermann Jahnke Dirk Simons	A rationale for the payback criterion
08-14	Peter Dürsch Jörg Oechssler Radovan Vadovic	Sick Pay Provision in Experimental Labor Markets
08-13	Carsten Schmidt Martin Strobel Henning Oskar Volkland	Accuracy, Certainty and Surprise - A Prediction Market on the Outcome of the 2002 FIFA World Cup
08-12	Mathias Sommer	Understanding the trends in income, consumption and
08-11	Hans Gersbach Hans Haller	Club Theory and Household Formation
08-10	Michael F. Meffert Thomas Gschwend	Strategic Voting in Multiparty Systems: A Group Experiment
08-09	Jens Wüstemann Jannis Bischof	Ausweis von Finanzinstrumenten in europäischen Bankbilanzen nach IFRS: Normative Erkenntnisse empirischer Befunde
08-08	Jürgen Eichberger David Kelsey	Are the Treasures of Game Theory Ambiguous?
08-07	Jürgen Eichberger Ani Guerdjikova	Multiple Priors as Similarity Weighted Frequencies