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Not a Fallacy of the Law of Large Numbers: Pooling Risks and the Utility of Insurance

PETER ALBRECHT

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By PETER ALBRECHT*

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Risk pooling involves the possibility of embedding a risk considered to be transferred to an insurance company in a collective of independent and identically distributed risks. We consider mutual insurers and stock insurance companies as well as alternative possibilities for setting the premium. Making standard assumptions with respect to the utility function and the distribution of claims, we are able to demonstrate that pooling risks on the side of the insurance company is beneficial for the potential policyholder in terms of expected utility. (JEL D 81)

*University of Mannheim, Schloss, D-68131 Mannheim, Germany (e-mail: risk@bwl.uni-mannheim.de). We would like to thank Nadine Gatzert and Hato Schmeiser for their fine paper on the merits of risk pooling, which has inspired our own analysis of the subject. We thank Markus Huggenberger, Raimond Maurer, Alexandr Pekelis, Hato Schmeiser and Patrick Schneider for their detailed and useful comments on draft versions of the paper.

Considering a sequence of independent and identically distributed (IID) replications of a gamble that is initially rejected, Samuelson (1963) famously demonstrates the "fallacy of large numbers" in a situation involving the summation of risks. As Ross (1999, p. 324) remarks, "Samuelson points out that the law of large numbers applies to *averages* and not to sums". Samuelson (1963) also notes that a situation that involves summing risks corresponds to an insurance company pooling many risks together. Finally, Samuelson (1963) emphasizes that it is not sufficient to merely consider the probability of loss (which is typically decreasing when the number of IID risks increases). Instead, it is the expected utility of wealth that is central to the evaluation.

In this paper, we examine the position of a potential buyer of an insurance contract instead of the position of an insurance company. Using only standard assumptions, we are able to demonstrate that risk pooling (on the side of the insurance company) is beneficial for the (potential) insured in terms of expected utility.

Embedding a risk in a collective consisting of n IID risks means that a single risk has a share of 1/n of the collective risk. Thus, it is indeed the law of large numbers (LLN) that is central to this effect[1]. We conclude that pooling risks is fundamental to the decision of purchasing insurance because it favorably affects the utility position of potential insurance buyers (policyholders), and we denote this phenomenon the "utility-improving effect of risk pooling".

Pooling risks together is considered the essence of insurance, and numerous papers[2] have analyzed the effects of risk pooling[3]. However, research on the effects of risk pooling is typically performed from the perspective of the insurance company, i.e., the supply side. Only recently have Gatzert and Schmeiser (2012) raised the question of whether risk pooling is also beneficial for policyholders, i.e., the demand side. However, they limit their analysis to a basic case (normally distributed claims, exponential utility function, mutual insurance) and do not perform a systematic analysis of the subject[4]. To the best of our knowledge, there are no further studies on the relevance of risk pooling from the perspective of potential

[1] By contrast, when analyzing the summation of risks, it is the central limit theorem or (more generally) the theory of large deviations - see, e.g., Brockett (1983) and Hammarlid (2005) - that are central to the analysis.
[2] For an excellent review of the corresponding literature, cf. Gatzert and Schmeiser (2012).
[3] However, these papers - in conformity with the common requirements of external regulation - are typically based on analyses of the probability of loss (probability of insolvency) and not on expected utility theory.
[4] In addition, they do not apply the probabilistic techniques of the present paper.

buyers of insurance. Neither in the theory of insurance demand[5] nor in the wider field of insurance economics[6] has this aspect of risk pooling attracted attention.

Related to the literature that addresses the effects of risk pooling from the perspective of the insurer is the strand of literature[7] that analyzes sequences of gambles in the tradition of Samuelson. However, the latter addresses a scenario (summation of risks instead of averages of risks; the default call, which is considered in section III, is not taken into account; nor is the legal form of the insurance company or the premium calculation principle applied) that differs from that analyzed in the present paper.

I. The Basic Decision Situation

We consider a decision maker that possesses an initial (t = 0) amount of wealth W_0. The potential accumulated claim $X \geq 0$ in t = 1 from a certain type of insurance is the basic random variable considered. If we assume a risk-free interest rate of r = 0 (interest is ignored) and that the decision maker does not buy insurance protection, the resulting end-of-period wealth (t = 1) is[8]

(1) $$W^{NI} = W_0 - X.$$

The alternative is to buy insurance protection for an individual premium[9] π ($\pi \leq W_0$). To simplify[10] the scenario, we assume that full coverage is available so that we do not have to make a distinction between the original claim and the indemnity. We assume that the risk X can be embedded in an IID collective of size n, i.e., in a homogeneous collective consisting of n independent risks[11] X_i (i = 1,…, n), $X_i \sim (\text{IID})X$. With respect to X, we assume that the parameters of the distribution of X are known[12] and that the expected value E(X) is finite[13].

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[5] See the recent review of Schlesinger (2013).
[6] See, for example, the Handbook of Insurance edited by Dionne (2013).
[7] Most notably, Diamond (1984), Nielsen (1985), Lippman and Mamer (1988), Hellwig (1995), Ross (1999) and Hammarlid (2005).
[8] We therefore ignore any additional insurable or non-insurable risks of the decision maker, particularly background risk - see Schlesinger (2013, pp. 180 ff.) - as well as any additional income (labor income, investment income, and/or pension claims).
[9] Understood as the total premium (including expenses and the profit margin), in contrast with the pure risk premium (premium for the risk transfer).
[10] This assumption is not a restriction. Defining $Y = h(X)$ as the insured part of the original risk X, the corresponding analysis must be based on Y.
[11] To keep the notation simple, we assume that the original risk X now corresponds to one of the risks X_i.
[12] Thus, we ignore parameter uncertainty.

In addition, we make the simplifying assumption that the IID-collective is the total collective of the insurance company because in this situation, the risk capital as well as the losses and gains of the company can be uniquely attributed to that collective.

The resulting end-of-period wealth (t = 1) in the case of purchasing insurance protection is $W^I(n)$. This position depends on additional factors (the legal form of the insurance company and the applied premium principle), and its concrete specification therefore has to be postponed.

The evaluation of the alternative wealth positions is performed in an expected utility framework; i.e., depending on the utility function, u(x), we have

(2a) $$\Phi^{NI} := \Phi(W^{NI}) = E[u(W^{NI})],$$

and

(2b) $$\Phi^I(n) := \Phi(W^I(n)) = E[u(W^I(n))].$$

According to (2b), the expected utility of the insured position depends, in particular, on the size of the IID collective in which the risk X is embedded.

With respect to the utility function, we initially make only standard assumptions. We assume that the utility function is strictly increasing and strictly concave. These assumptions ensure the continuity[14] of the utility function and the existence of the inverse function u^{-1}, which again is strictly increasing. In addition, we make the normalization u(0) = 0. Moreover, as any concave function is dominated from above by an affine function, the existence of E(X) ensures the existence of E[u(X)] in the sense of Klenke (2014, Definition 4.7), i.e., as an extended real number in the domain $[-\infty, \infty)$. This property carries over to the quantities $E[u(W^{NI})]$ and $E[u(W^I(n))]$ introduced in (2).

With respect to the utility effects of pooling risks, we consider two versions. The first ("strong") version requires that the expected utility of the corresponding wealth position be strictly increasing in the size of the collective. The second ("weak") version does not require the effects to be strictly monotone.

In the strong version, we require the following two conditions for a utility-improving effect of risk pooling:

[13] As $X \geq 0$, we also have $E(|X|) < \infty$; i.e., X is absolutely integrable. Combined with the IID assumption, this ensures the validity of the strong LLN.
[14] In case the domain of u(x) is an interval that is closed on the left, we have to postulate continuity separately for this left end point.

(3a) $\Phi^{I}(2) > \Phi^{NI}$

(3b) $\Phi^{I}(n+1) > \Phi^{I}(n)$ for all $n \geq 2$.

Condition (3a) indicates that merely embedding the risk in an IID collective of size two already leads to an improvement in expected utility, and that purchasing insurance will be preferred. In addition, we require that the improvement in expected utility strictly increases with the size of the collective.

In the weak version, purchasing insurance, i.e., embedding the risk in an IID collective, is beneficial for the decision maker if there exists a size n_0 of the collective with

(4a) $\Phi^{I}(n) > \Phi^{NI}$ for all $n \geq n_0$.

Stating (4a) alternatively in terms of the certainty equivalent, we require $CE^{I}(n) > CE^{NI}$, where $CE^{I}(n) \coloneqq u^{-1}(\Phi^{I}(n))$ and $CE^{NI} \coloneqq u^{-1}(\Phi^{NI})$. Condition (4a) is obviously fulfilled if we are able to demonstrate that $\lim_{n\to\infty} \Phi^{I}(n) > \Phi^{NI}$ or, equivalently, $\lim_{n\to\infty} CE^{I}(n) > CE^{NI}$.

A utility-improving effect from increasing the size of the collective holds in the weak version if for all $n \geq n_0$, there is a collective size $k_n > n$ with

(4b) $\Phi^{I}(k) \geq \Phi^{I}(n)$ for all $k \geq k_n$.

The existence of a limit $\lim_{n\to\infty} \Phi^{I}(n) > \Phi^{NI}$ or, equivalently, a limit $\lim_{n\to\infty} CE^{I}(n) > CE^{NI}$ obviously ensures the fulfillment of condition (4b) as well.

Condition (4a) indicates that for a given risk X and a given utility function u(x) there exists an IID collective $X_i \sim (IID)X$ of a minimum size n_0, such that transferring the risk to the insurer will always be beneficial to a decision maker with utility function u. Condition (4b) indicates that embedding the risk in a (sufficiently) larger collective leads to an additional improvement in expected utility.

In both versions, the possibility of embedding a risk (considered to be transferred to an insurance company) in an IID collective ensures that purchasing insurance is (at least ultimately) utility improving.

II. Pure Mutual Insurance

In the case of pure mutual insurance, the policyholders are also the owners of the insurance company. Thus, policyholders have to participate in the company's profits and losses. To simplify the analysis, we assume that the insurance company is able to distribute possible profits and losses completely[15]. With a homogeneous collective, it is natural to assume that every policyholder will have an equal share of the profits and/or the losses of the collective[16].

We let $S_n := X_1 + \ldots + X_n$ denote the accumulated claim of the IID collective and $\bar{S}_n := S_n/n$ represent the average accumulated claim per insured. If we assume that all members of the homogeneous collective are charged an identical individual premium π, the wealth position of the decision maker - whose risk is embedded in an IID collective of size n - is given by

(5) $$W^I(n) = W_0 - \pi + (n\,\pi - S_n)/n = W_0 - \bar{S}_n.$$

In the case of pure mutual insurance, therefore, we have a special situation in which the wealth position of the insured is (under the specified assumptions) completely independent of the charged individual premium π. Because the policyholders have to take an equal share in all profits and losses, the amount of the premium becomes irrelevant.

We are now able to formulate our central result regarding the utility-improving effects of risk pooling in the case of pure mutual insurance.

THEOREM 1: *The wealth position (5) in the case of pure mutual insurance fulfills the requirements (3) of a utility-improving effect of risk pooling for all potential buyers of insurance with a utility function that is strictly increasing and strictly concave.*

This result is important not only from the perspective of the legal form of the insurance company. Modern insurance is rooted in mutual insurance, and the essence of mutual insurance is the notion that re-distributing risk in a collective is more beneficial than assuming the risk alone. Theorem 1 offers a utility-based justification of this classical notion.

The proof of Theorem 1 is given in the appendix. In the following, we sketch the main lines of the argument. The idea of Theorem 1 is to show that the wealth position W^{NI} is "riskier" than $W^I(2)$ and that the positions $W^I(n)$ are in each case also "riskier" than the positions

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[15] Thus, we ignore taxes on profits as well as the possibility of increasing the reserves of the company.
[16] Thus, we exclude the possibility that the insurance company is taking out a loan to finance the losses.

$W^I(n+1)$. Here, "riskier" is understood as one of the equivalent versions (to be specific: Z = Y + "Noise") of the concept of "increasing risk" according to Rothschild and Stiglitz (1970). Defining $Z = W^I(n)$ and $Y = W^I(n+1)$, we obtain $\varepsilon = \bar{S}_{n+1} - \bar{S}_n$ as the relevant noise quantity. Verifying the condition $E(\varepsilon|Y) = 0$ requires the evaluation of the quantities $E(\bar{S}_n|\bar{S}_{n+1})$. For this, we exploit the result that the sequence $\{\bar{S}_n\}$ is a backwards martingale[17] (also known as a reverse martingale).

Finally, we illustrate[18] the results of Theorem 1 by assuming normally distributed IID risks X_i, i.e., $X \sim N(\mu, \sigma^2)$, and an exponential utility function, which is consistent with using the preference functional ($a > 0$)

(6) $$\Phi(X) = E(X) - a\, Var(X).$$

We have $\Phi^{NI} = \Phi(W_0 - X) = E(W_0 - X) - a\, Var(W_0 - X)$, and therefore, on one hand, we obtain

(7) $$\Phi^{NI} = W_0 - \mu - a\, \sigma^2.$$

On the other hand, we have $\bar{S}_n \sim N(\mu, \sigma^2/n)$ and $\Phi^I(n) = E(W_0 - \bar{S}_n) - a\, Var(W_0 - \bar{S}_n)$ and obtain

(8) $$\Phi^I(n) = W_0 - \mu - a\, \sigma^2/n.$$

Obviously, the relations $\Phi^I(2) > \Phi^{NI}$ and $\Phi^I(n+1) > \Phi^I(n)$ for all $n \geq 2$ are valid in this situation, which confirms the results of Theorem 1 in one special case (the combination of a distributional assumption, on the one hand, with an assumption regarding the utility function, on the other hand).

III. Stock Insurance Company

A. Basic Considerations

We consider an insurance company with an initial ($t = 0$) risk capital of amount C. The insurance company will become (technically) insolvent if $S_n > n\, \pi + C$; i.e., the accumulated

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[17] For this result, see, for example, Klenke (2014, section 12.2).
[18] This illustration corresponds to the basic case that is considered in Gatzert and Schmeiser (2012).

claim amount of the collective exceeds the sum of the accumulated premiums and the risk capital at hand.

As a consequence of insolvency, the resulting loss amount $L_n = \max(S_n - n\,\pi - C, 0)$ must be borne by the collective of the insured. Given this situation, we analyze two basic issues. First, we analyze the behavior of the relative loss[19] $\bar{L}_n = L_n/n$, i.e., the average loss per member of the IID collective. We have

(9) $$\bar{L}_n = \max(\bar{S}_n - \pi - C/n, 0).$$

From the perspective of a single insured, the quantity $-\bar{L}_n$ is of relevance, because the relative loss reduces the wealth position. This quantity can be regarded as a short position in a call option (a "default call"). Transferring the risk to a stock insurance company implies that (per capita) the policyholder takes a short position in this default call. The call option is defined relative to the average loss variable $\bar{S}_n$ and has an exercise price of amount $\pi + C/n$. As there exists no market in which average loss variables (and also absolute loss variables) are traded and as it is not possible to generate the option on the basis of a self-financing dynamic trading strategy, we do not consider using arbitrage-free valuation techniques to price the option. Consistent with our general valuation approach, we value the call using expected utility theory.

In a second step, we consider the indemnity of the policyholder in case of (technical) insolvency. In this case, we have to adjust the position of policyholder i in proportion to the amount of his loss X_i. The resulting wealth position[20] is now[21] $W^I(n) = W_0 - \pi - L_n \cdot (X_i/S_n)$ or, equivalently,

(10) $$W^I(n) = W_0 - \pi - \bar{L}_n \cdot (X_i/\bar{S}_n).$$

We have $\sum_{i=1}^{n} L_n \cdot (X_i/S_n) = L_n$; i.e., on the level of the collective, we correctly obtain the loss L_n that results from the insolvency of the insurance company.

[19] Simultaneously, this equals the loss, which every policyholder must bear in the case of pure mutual insurance.
[20] Obviously, there exists no elementary method to evaluate this position. Thus, we must resort to more fundamental probabilistic arguments.
[21] We define $X_i/S_n = 0$ in the case of $X_i = 0$. In the case of $X_i > 0$, we have $S_n > 0$, so there is no problem with the term X_i/S_n.

To simplify the following considerations, we additionally assume[22] $X \leq M < \infty$ in this section, which indicates that the individual accumulated claim cannot be arbitrarily high. From the perspective of a practical application, this assumption is not problematic.

This additional assumption is motivated by the fact that the proofs of the following theorems require the interchangeability of integration (calculating the expected value) and almost sure convergence. This requirement is assured by Lebesgue´s dominated convergence theorem[23], which requires, in turn, the existence of an integrable dominating function. However, a strictly concave utility function u cannot be bounded from below over an infinite domain. To avoid complex restrictions[24] on the behavior of the utility function and/or the claim distribution in the left tail, the assumption of a finite domain of X must be made from the outset[25]. In addition, it must be ensured that the utility function only takes finite values at the endpoints of the domains of the wealth positions W^{NI}, $W^{I}(n)$ and $\overline{L}_n$. As becomes apparent from the proofs of the theorems in the Appendix, it is, for example, sufficient to assume $u(W_0) < \infty$ and $u(-M) > -\infty$. In addition, these assumptions ensure that the positions in (2), i.e., the expected utility of the wealth positions, assume finite values.

While we need to introduce an additional requirement for the risk variable X in this section, the requirements with respect to the utility function u(x) can be weakened. All wealth positions analyzed in this section, i.e. W^{NI} , $W^{I}(n)$ and $\overline{L}_n$, have a domain of definition which is either identical to the interval $(-\infty, W_0]$ or to a subset of this interval. Thus, it is sufficient to restrict the requirements of strict monotonicity and strict concavity of u(x) to the domain $(-\infty, W_0]$. Outside this domain the utility function may have an arbitrary shape. For instance, the utility function may be of the Friedman/Savage type, combining risk aversion on the domain $(-\infty, W_0]$ and risk seeking on the domain (W_0, ∞).

In the remaining sections, we analyze the utility-improving effects of risk pooling by assuming two common alternative basic possibilities of setting the premium.

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[22] Concurrently, this is a standard assumption in the theory of insurance demand; see Schlesinger (2013, p. 168).
[23] See, for example, Klenke (2014, Corollary 6.26).
[24] See, for instance, Lippman and Mamer (1988), who analyze sequences of gambles in the tradition of Samuelson.
[25] Such an assumption is quite common. Föllmer and Schied (2011, p. 70 resp. p. 77), for example, make this assumption when analyzing the optimality of alternative insurance contracts resp. sequences of gambles.

B. *Insurance Premiums Independent of the Size of the Collective*

We first consider the case in which the individual insurance premium π is independent of the size n of the IID collective of policyholders. We disaggregate the individual premium in the form $\pi = \mu + l$, where l denotes the loading to the net risk premium $\mu = E(X)$, making the standard assumption $l \geq 0$. Loading l consists of a safety loading, on the one hand, and a loading for expenses, on the other.

As our first result, we obtain:

THEOREM 2: *Considering a stock insurance company and an individual premium $\pi \geq E(X)$, which is independent of the size of the IID collective insured, we obtain $E\left[u\left(-\bar{L}_n\right)\right] \to 0$ for $n \to \infty$ under the assumptions specified*[26].

Thus, on a per capita basis, the value (in expected utility terms) of the short position in the default call, i.e., the "disutility" implied by the potential default of the insurance company, converges to zero when the size of the IID collective grows beyond all limits. If the risk to be transferred to the insurance company can be embedded in an IID collective of sufficient size, the possibility of the default of the insurance company (on average) is no longer relevant to the decision to purchase insurance.

The proof of Theorem 2 is presented in the appendix. In the following, we present the main lines of argument. From the strong LLN, we first have $\lim_{n\to\infty} \bar{S}_n = \mu$ almost surely (a.s.). Applying continuity arguments, we then obtain $-\bar{L}_n \to 0$ a.s. and therefore also $u\left(-\bar{L}_n\right) \to u(0) = 0$ a.s. In the next step, we have to show the relation $E\left[u\left(-\bar{L}_n\right)\right] \to E[u(0)]$. This step, however, is not trivial because it is well known that almost sure convergence does not imply[27,28] convergence in expectation (resp. L^1-convergence). To prove the necessary convergence relation, we identify a dominating function that enables us to use Lebesgue´s dominated convergence theorem. The appendix contains additional details.

We now evaluate the wealth position (10). For the original position of the decision maker (no purchase of insurance protection), we have $\Phi(W^{NI}) = E[u\,(W_0 - X)]$. In case of risk

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[26] Analyzing the proof of Theorem 2, we de facto only require that the utility function is increasing and continuous.

[27] See, for instance, Klenke (2014, Remark 6.11).

[28] Regarding the sequence $\{\bar{S}_n\}$, L^1-convergence is ensured, as $\{\bar{S}_n\}$ is a backwards martingale and therefore uniformly integrable. However, this does not necessarily translate to the sequence $\{u(-\bar{L}_n)\}$.

aversion[29], we obtain $CE^{NI} = u^{-1}[\Phi(W^{NI})] < E(W^{NI}) = W_0 - E(X) = W_0 - \mu$ for the safety equivalent. Therefore, we have

$$\text{(11)} \qquad CE^{NI} < W_0 - \mu.$$

For a given risk X and a given utility function u(x) we therefore define the quantity $\Delta := |W_0 - \mu - CE^{NI}| > 0$.

THEOREM 3: *We consider an individual premium $\pi \geq E(X)$, which is independent of the size of the IID collective insured. Under the assumptions specified, the wealth position (10) in the case of a stock insurance company fulfills the requirements (4) for a utility-improving effect of risk pooling if $l < \Delta$.*

The proof of Theorem 3 is presented in the appendix and follows the lines of the proof of Theorem 2. The additional requirement $l < \Delta$ can now be explained as follows. From the proof of Theorem 3, we obtain $\lim_{n\to\infty} \Phi^I(n) = u(W_0 - \pi)$ in the first step. To fulfil relation (4a), we must additionally establish $u(W_0 - \pi) > \Phi^{NI}$ or equivalently $W_0 - \pi > CE^{NI}$. For a risk-averse policyholder, we have $W_0 - \mu > CE^{NI}$, as previously discussed. The additional requirement $l < \Delta$ now ensures that we also have $W_0 - \pi > CE^{NI}$.

The requirement $l < \Delta$ implies a restriction with respect to the amount of the premium, π, charged. Only if this restriction is valid does the possibility of pooling risk on the side of the insurance company remain beneficial for the potential buyer of insurance protection. An additional increase in the individual premium reduces the probability of insolvency and favorably influences the value of the default call $\max(\bar{S}_n - \pi - C/n, 0)$. However, with respect to the total utility of the position (10), this effect is overcompensated by the negative effect of a higher premium π, which indicates that the pooling of risks is no longer beneficial in the case in which the insurance premium is “too high”.

A case of special interest is $l = 0$ (zero loading); i.e., the premium corresponds to the expected claim amount (actuarially fair premium), $\pi = \mu$. In this case[30], the probability of insolvency of the insurance company is $P(S_n > n\,\mu) = P[(S_n - n\,\mu)/(\sigma/\sqrt{n}) > 0)$. From the Central Limit Theorem, we obtain that this probability converges to (N denotes the distribution function of the standard normal distribution) $1 - N(0) = ½$ for $n \to \infty$. Therefore,

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[29] According to our assumptions, the utility function u(x) is strictly concave. Assuming that the random variable X is not degenerated (not a constant quantity), we employ Jensen´s inequality to ensure that the safety equivalent is strictly less than the expected value.

[30] Making the additional simplifying assumption C = 0.

the probability of insolvency does not converge to zero for IID collectives growing in size beyond all limits but "stabilizes" near ½.

Intuitively, we may conjecture that this result is relevant for the evaluation of the wealth positions (9) and (10) of the potential buyer of insurance protection, because the preceding result may have an influence on the value of the default call $\overline{L}_n = \max(\overline{S}_n - \pi, 0) = \max(S_n - n\,\mu, 0)/n$. However, Theorem 2 and Theorem 3 (for l = 0, the condition l < Δ is obviously valid) demonstrate that this conjecture is not true when the wealth positions are evaluated using expected utility. This result clearly shows that the probability of insolvency of the insurance company, on the one hand, and the expected utility of the wealth positions (9) and (10) of the (potential) policyholder, on the other hand, do not exhibit a direct relation.

We now present an illustration of the preceding results, again assuming the setting considered in the illustration of Theorem 1 in section II (normally distributed risks and exponential utility function). As the evaluation of the wealth position (10) in the form of a closed formula is not possible even in this most basic case, we confine ourselves to evaluating position (9).

To simplify the analysis[31], we assume that the premium is actuarially fair, $\pi = \mu$; i.e., the premium corresponds to the net risk premium (expected claim amount) and, in addition, a risk capital of amount zero, i.e., C = 0. According to section 2 of the online appendix, we obtain

(12a) $$E(\overline{L}_n) = n(0)\,\sigma/\sqrt{n},$$

(12b) $$Var(\overline{L}_n) = [0.5 - n(0)^2]\,\sigma^2/n$$

in this case, where n(x) denotes the density function of the standard normal distribution. We have $n(0) \approx 0.4$ and $[0.5 - n(0)^2] \approx 0.341$.

Obviously, we have $\lim_{n\to\infty} E(\overline{L}_n) = 0$ and $\lim_{n\to\infty} Var(\overline{L}_n) = 0$. In addition, $E(\overline{L}_n)$ and $Var(\overline{L}_n)$ are strictly decreasing with n, which confirms the conclusion of Theorem 2 for this special case; as with $E\left[u(-\overline{L}_n)\right] = a\,Var(\overline{L}_n) - E(\overline{L}_n)$, we obtain $\lim_{n\to\infty} E[u(-\overline{L}_n)] = 0$. In addition, this

[31] The evaluation of the term max $(\overline{S}_n - \pi - C/n, 0)$ in (9) requires the calculation of partial moments, which is undertaken in the online appendix. In the general case, the calculation leads to rather complicated results. For this reason, we confine the presentation to a basic case.

result confirms the preceding general analysis, according to which in the case $\pi = \mu$ as well, there is a utility-improving effect of risk pooling for the potential buyer of insurance.

At the same time, this result shows that the assumption of a finite domain of X can be dispensed with in special constellations. This is particularly the case when X is normally distributed, as expected utility then ultimately only depends on the expected value and the variance of the quantity to be evaluated.

C. Insurance Premiums Dependent on the Size of the Collective

Smith and Kane (1994) and Gatzert and Schmeiser (2012) distinguish between two effects of pooling risks. In case B, the probability of insolvency is decreasing (in the limit to a value of zero) when the individual premium is fixed. In case A, the probability of insolvency is fixed at a tolerated level, and the resulting premium is decreasing (in the limit to the expected claim amount; i.e., the loading converges to zero) with the increasing size of the insured IID collective. In case A the risk-pooling effect of case B is “passed through” to the policyholders in the form of a premium reduction as long as the required safety level is ensured. The constellation of case B is analyzed in section III.B. In the present section, we analyze a situation corresponding to case A.

In case A, risk pooling leads to premiums that are subadditive. A utility theoretic analog is considered in Diamond (1984, pp. 405 ff.)[32]. According to Diamond, summing independent risks provides diversification when the premium per risk decreases.

In insurance mathematics, the property of subadditivity is regarded essential for calculating premiums[33]. Shaun Wang et al. (1997), therefore, employ an axiomatic approach to characterizing insurance premiums in a competitive market setting. Their approach leads to a price functional of the Choquet type, which in turn implies subadditive premiums.

[32] Additional arguments with respect to this link to subadditivity are provided by Denuit et al. (2011, p. 243 f.).

[33] Analogously, the requirement of subadditivity is one of the central axioms for a coherent risk measure (which relates to risk capital instead of risk premiums) in the prominent axiomatization presented by Artzner et al. (1999).

Subadditive premiums have been criticized[34] for not being consistent with the fundamental principle of no arbitrage, because this principle implies linear price functionals. However, this criticism is valid only for markets without frictions and is no longer valid for markets with frictions (such as transaction costs[35] or trading). De Waegenaere et al. (2003) demonstrate that Choquet pricing is consistent with general equilibrium in the case of insurance markets[36] with frictions.

Consistent with the preceding discussion, in the following, we consider an individual premium of the form

(13a) $$\pi = \mu + l(n),$$

where $l(n) \geq 0$ is a loading (per risk) that depends on the size of the IID collective[37]. In addition, we are demanding

(13b) $$\lim_{n\to\infty} l(n) = 0.$$

In contrast to section III.B, the individual premium now depends on the size of the collective, i.e., $\pi = \pi(n)$.

First, we note that the result of Theorem 2 also holds in the case in which the premium depends on the size of the collective. The disutility of the short position in a default call resulting from a possible insolvency of the stock insurance company converges to zero with an increasing size of the collective of insured. The proof of this statement completely parallels the proof of Theorem 2. In addition, we have the following:

THEOREM 4: *We consider an individual premium of the form (13) that is dependent on the size of the IID collective insured. Under the assumptions specified, the wealth position (10) in the case of a stock insurance company fulfills the requirements (4) of a utility-improving effect of risk pooling.*

The proof of Theorem 4 is presented in the appendix and follows the lines of the proof of Theorem 3. However, contrary to Theorem 3, no additional restrictions with respect to loading l are necessary. Demanding $l(n) \to 0$ is already sufficient to ensure the fulfillment of the requirements in (4).

[34] See, for example, Gatzert and Schmeiser (2012, p. 188, p. 190). Borch (1982, p. 1295) previously stated the following: "If a competitive market is in equilibrium, values must be additive…".
[35] Gollier (2013, p. 118) reports that transaction costs in the case of insurance markets are approximately 30% of the premium, which is significantly larger than in the case of financial markets.
[36] A corresponding result for financial markets is obtained by Chateauneuf et al. (1996).
[37] This loading, for example, consists of a declining safety loading per insured, on the one hand (see, also, the following example), and regressive operating costs, on the other hand.

The result of Theorem 4 in our view is fundamental to the understanding of the beneficial aspects of risk pooling for the (potential) policyholders. Embedding the risk to be transferred in an IID collective that is large enough *and* charging a premium of the form (13) will always be preferable to not insuring the risk!

We end our discussion with an illustrative example, again confining ourselves to the evaluation of the wealth position (9). We consider an IID collective of normally distributed risks in connection with the preference functional (6). To simplify the analysis, we assume C = 0 and ignore expenses. We now fix the level α $(0 < \alpha < 0{,}5)$ of the probability of insolvency, i.e., $P(S_n > n\pi(n)) = \alpha$. Per definition, the quantity $n\pi(n)$ must be identical to the (1-α)-quantile of the distribution of S_n. Because S_n is normally distributed, we therefore obtain the condition $n\pi(n) = E(S_n) + N_{1-\alpha}\,\sigma(S_n) = n\mu + N_{1-\alpha}\sigma\sqrt{n}$, where $N_{1-\alpha}$ denotes the (1-α)-quantile of the standard normal distribution, which implies that the structure of the (individual) safety loading has the form

(14) $$l(n) = N_{1-\alpha}\sigma/\sqrt{n}.$$

Thus, the safety loading exhibits an inverse proportional relationship to the square root of the size of the collective.

Charging the premium $\pi(n) = \mu + l(n)$, we obtain[38]

(15a) $$E(\overline{L}_n) = h_1(\alpha)\,\sigma/\sqrt{n}\,,$$

(15b) $$Var(\overline{L}_n) = h_2(\alpha)\,\sigma^2/n.$$

The functions $h_1(\alpha) > 0$ and $h_2(\alpha) > 0$ are strictly positive and depend only on the probability level α but do not depend on the size of the collective. This finding confirms the conclusion of Theorem 2 for this special case; because $E\left[u(-\overline{L}_n)\right] = a\,Var(\overline{L}_n) - E(\overline{L}_n)$, we obtain $\lim_{n\to\infty} E[u(-\overline{L}_n)] = 0$.

[38] As in the example from section III.B, the evaluation of position (9) requires the calculation of partial moments. The calculations are contained in the online appendix.

IV. Concluding Remarks

In this paper, we have analyzed the relevance of risk pooling on the side of the insurance company for potential buyers of an insurance contract. Risk pooling involves the possibility of embedding a risk that is considered to be transferred to an insurance company in a collective of independent and identically distributed risks (an “IID collective”). We distinguish the case of a pure mutual insurer from the case of a stock insurance company. In the latter, the policyholder is formally exposed to a short position in a “default call”; i.e., the policyholder has to include the effects of the insurance company’s potential insolvency in his evaluation. In addition, the alternative basic possibilities of setting the premium (independent of or dependent on the size of the collective) are considered.

We assume that a potential customer of the insurance company evaluates his alternative wealth positions based on expected utility. On the basis of the strong law of large numbers and elementary probabilistic arguments, we are then able to demonstrate - making only standard assumptions with respect to the utility function and the claim distribution - that the pooling of risks on the side of the insurer is beneficial for the (potential) policyholder. In addition, sufficiently increasing the size of the IID collective leads to a further increase in utility.

Thus, we conclude that the possibility of risk pooling is essential to the decision to purchase insurance because it favorably affects the utility position of the potential buyer of insurance.

The utility-based analysis of embedding a risk that is considered to be transferred to an insurance company in a collective of risks demonstrates that insurance contracts exhibit significant differences compared to other financial products. This finding confirms the view[39] that “producing” insurance is based on a “production law” (i.e., pooling risks) sui generis.

[39] See, for example, Gatzert and Schmeiser (2012, p. 184).

Appendix: Proofs

Theorem 1:

We exploit the fact that the sequence $\{\bar{S}_n; n \geq 1\}$ is a backwards martingale (reverse martingale)[40]. From the foregoing, one can deduce[41] the validity of the following relation $(n \geq 1)$

(A.1) $\quad E(\bar{S}_n | S_{n+1}) = \bar{S}_{n+1},$

which is central to our proof. Because the information sets generated by $\{S_{n+1}\}$ and $\{\bar{S}_{n+1}\}$ are identical, we also have $E(\bar{S}_n | \bar{S}_{n+1}) = \bar{S}_{n+1}$ for $(n \geq 1)$.

We now define $Z \coloneqq W_0 - \bar{S}_n$ and $Y \coloneqq W_0 - \bar{S}_{n+1}$. We then have $Z = Y + \varepsilon$ with $\varepsilon = \bar{S}_{n+1} - \bar{S}_n$. From $E(\bar{S}_{n+1} | \bar{S}_{n+1}) = \bar{S}_{n+1}$ and (A.1), it follows that $E(\varepsilon | W_0 - \bar{S}_{n+1}) = E(\varepsilon | \bar{S}_{n+1}) = E(\bar{S}_{n+1} | \bar{S}_{n+1}) - E(\bar{S}_n | \bar{S}_{n+1}) = \bar{S}_{n+1} - \bar{S}_{n+1} = 0$.

Therefore, for all $n \geq 1$, $W_0 - \bar{S}_n$ is "riskier" than $W_0 - \bar{S}_{n+1}$ in the sense of the definition of Rothschild and (1970), i.e., $W_0 - \bar{S}_n = (W_0 - \bar{S}_{n+1})$ + "Noise". Thus, we have $E[u(W_0 - \bar{S}_{n+1})] \geq E[u(W_0 - \bar{S}_n)]$ for all risk-averse decision makers, which corresponds to requirements (3).

Theorem 2:

We proceed from $\pi = \mu + l$ with $l \geq 0$. The functions $f(x) = \max(x,0)$, $g(x) = \min(x,0)$ and $u(x)$ are continuous. Thus, from $\bar{S}_n \rightarrow \mu$ almost surely (a.s.) and $C/n \rightarrow 0$ (surely), we obtain $-\bar{L}_n = -\max\left(\bar{S}_n - \mu - l - C/n, 0\right) = \min\left(\mu + l + C/n - \bar{S}_n, 0\right) \rightarrow \min(l,0) = 0$ a.s. and, therefore, also $u(-\bar{L}_n) \rightarrow 0$ a.s. With $0 \leq X \leq M$, we have $0 \leq \bar{S}_n \leq M$, and therefore, we obtain $-M < \min\left(\pi + C/n - \bar{S}_n, 0\right) \leq 0$. Altogether, given our assumptions, we obtain $|u(-\bar{L}_n)| \leq |u(-M)| < \infty$. Thus, we have identified a dominating function for $|u(-\bar{L}_n)|$. The dominating function $|u(-M)|$ is constant and therefore (absolutely) integrable. Thus, we can

[40] See, for example, Klenke (2014, Chapter 12.2).
[41] See Föllmer and Schied (2011, relation (2.20)).

apply Lebesgue´s dominated convergence theorem[42] and obtain $\lim_{n\to\infty} E[u(-\bar{L}_n)] = E[\lim_{n\to\infty} u(-\bar{L}_n)] = 0$.

Theorem 3:

We follow the lines of the proof of Theorem 2. From $\bar{S}_n \to \mu$ a.s. and $\bar{L}_n \to 0$ a.s., we obtain $W^I(n) \to W_0 - \pi$ a.s., where $W^I(n) = W_0 - \pi - (\bar{L}_n/\bar{S}_n)X_i$. Therefore, we also obtain $u(W^I(n)) \to u(W_0 - \pi)$ a.s. In addition, we have $-M < W^I(n) \le W_0 - \pi < W_0$ and therefore $u(-M) \le u(W^I(n)) \le u(W_0)$. Overall, we obtain on the basis of our assumptions that $|u(W^I(n))| \le \max\left(u(W_0), |u(-M)|\right) < \infty$. Thus, we have found a dominating integrable function, and therefore, we can apply Lebesgue´s dominated convergence theorem, so we are able to conclude that $E[u(W^I(n))] \to E\left[u\left(W_0 - \pi\right)\right] = u(W_0 - \pi)$.

Theorem 4:

We proceed from $\pi(n) = \mu + l(n)$ with $l(n) \ge 0$. The result $\bar{L}_n \to 0$ a.s. of Theorem 2 holds as well in the case of $\bar{L}_n = \max(\bar{S}_n - \pi(n) - C/n, 0)$. From condition (13b), i.e. $\lim_{n\to\infty} l(n) = 0$, we thus obtain $W^I(n) \to W_0 - \mu$ a.s., in which $W^I(n) = W_0 - \pi(n) - (\bar{L}_n/\bar{S}_n)X_i$, and therefore, altogether $u(W^I(n)) \to u(W_0 - \mu)$ a.s. With respect to $|u(W^I(n))|$, we obtain the same dominating function as in Theorem 3. Thus, we finally obtain $E[u(W^I(n))] \to u(W_0 - \mu)$.

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[42] See, for instance, Klenke (2014, Corollary 6.26).

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O. Calculations for the Examples

O.1 Partial Moments of the Normal Distribution

Using the results of Winkler et al. (1972) on partial moments, we generally obtain the following for $Y \sim N(\mu, \sigma^2)$:

(O.1a) $$XE_z(Y) = \sigma\, n(m) - (z - \mu)\, N(-m)$$

(O.1b) $$XV_z(Y) = \sigma^2 N(-m) + (z - \mu)^2 N(-m) - \sigma(z - \mu)\, n(m),$$

in which $m = (z - \mu)/\sigma$. The quantities $N(x)$ resp. $n(x)$ denote the distribution function resp. the density function of the standard normal distribution, and the quantities $XE_z = E[\max(X - z, 0)]$ resp. $XV_z = E[\max(X - z, 0)^2]$ stand for the excess expectation resp. the excess variance with regard to a target quantity z.

For $\bar{L}_n = \max(\bar{S}_n - \pi, 0)$, we therefore have with $\bar{S}_n \sim N(\mu, \sigma^2/n)$, $z = \pi$, $\sigma = \sigma/\sqrt{n}$ and $m=m_n = \sqrt{n}\,(z - \mu)/\sigma$

(O.2a) $$E(\bar{L}_n) = XE_\pi(\bar{S}_n) = \sigma\, n(m_n)/\sqrt{n} - (z - \mu)N(-m_n)$$

In addition, we obtain

(O.3a) $$E\left(\bar{L}_n^2\right) = XV_\pi(\bar{S}_n) = \frac{\sigma^2}{n} N(-m_n) + (z - \mu)^2 N(-m_n) - \frac{\sigma}{\sqrt{n}}(z - \mu)\, n(m_n)$$

and therefore

(O.3b) $$\begin{aligned} Var(\bar{L}_n) = {} & \frac{\sigma^2}{n}[N(-m_n) - n(m_n)^2] \\ & - \frac{\sigma}{\sqrt{n}}(z - \mu)\, n(m_n)[1 - 2N(-m_n)] \\ & + (z - \mu)^2 N(m_n)N(-m_n). \end{aligned}$$

O.2 Actuarially Fair Premium

In this case, we have $\pi = \mu$, $z - \mu = 0$, $m_n = 0$, $N(m_n)\, N(-m_n) = N(0) = 0.5$ and $n(m_n) = n(0) = 1/\sqrt{2\pi} \approx 0.3989$. Therefore, we obtain

(O.4a) $$E(\bar{L}_n) = \sigma\, n(0)/\sqrt{n} \approx 0.3989\, \sigma/\sqrt{n}$$

(O.4b) $$\mathrm{Var}(\bar{L}_n) = \frac{\sigma^2}{n}[0.5 - n(0)^2] \approx 0.341 \cdot \frac{\sigma^2}{n}.$$

O.3 Safety Loading (14)

We have $z = \pi = \mu + N_{1-\alpha}\sigma/\sqrt{n}$, $z - \mu = N_{1-\alpha}\sigma/\sqrt{n}$, $m_n = N_{1-\alpha}$, $N(m_n) = N(N_{1-\alpha}) = 1 - \alpha$ and $N(-m_n) = \alpha$. We therefore obtain

(O.5a) $$E(\bar{L}_n) = \frac{\sigma}{\sqrt{n}}[n(N_{1-\alpha}) + (1 - \alpha)N_{1-\alpha}]$$

resp.

(O.5b) $$E(\bar{L}_n) = h_1(\alpha)\, \sigma/\sqrt{n}\,,$$

using the auxiliary quantity

(O.5c) $$h_1(\alpha) \coloneqq n(N_{1-\alpha}) + (1 - \alpha)\, N_{1-\alpha}.$$

Obviously, we have $h_1(\alpha) > 0$ for $\alpha < 0.5$.

In addition, we have

(O.6a) $$\mathrm{Var}(\bar{L}_n) = \frac{\sigma^2}{n}[\alpha - n^2(N_{1-\alpha}) - N_{1-\alpha}\, n(N_{1-\alpha})(1 - 2\alpha) + N_{1-\alpha}^2\alpha(1 - \alpha)]\,,$$

resp.

(O.6b) $$\mathrm{Var}(\bar{L}_n) = h_2(\alpha)\, \sigma^2/n\,,$$

using the auxiliary quantity

(O.6c) $$h_2(\alpha) \coloneqq N_{1-\alpha}^2\alpha(1 - \alpha) + \alpha - n^2(N_{1-\alpha}) - N_{1-\alpha}\, n(N_{1-\alpha})(1 - 2\alpha).$$

Again, we have $h_2(\alpha) > 0$ for $0 < \alpha < 0.5$ (which is confirmed by a plot of the function).

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