Contributions to the Theory of Solution Concepts for Strategic Games

Inauguraldissertation
zur Erlangung des akademischen Grades
eines Doktors der Wirtschaftswissenschaften
der Universität Mannheim

Alexander Zimper

vorgelegt im Sommersemester 2003

Referent:	Prof. Martin Hellwig, Ph.D.
Korreferent	Prof. Itzhak Gilboa, Ph.D.
Dekan:	Prof. Dr. Christoph Buchheim
Tag der mündlichen Prüfung:	21. Juli 2003

Acknowledgments

My thesis consists of 3 ¼ self-contained papers of which older versions had been circulating around for some time. ‘Circulating around’ means: I bothered some people to read them and surprisingly many took their time to give me valuable comments. I would like to thank the following people for their contributions:

Tilman Börgers, Eddie Dekel, Christian Ewerhart, Itzhak Gilboa, Bertrand Koebel, and Ulrich Schmidt.

As a member of the Mannheimer Graduiertenkolleg "Allokation auf Finanz-und Gütermärkten" I received financial support from the Deutsche Forschungsgemeinschaft. Furthermore, while I stayed three months at the University College of London, a Marie-Curie-scholarship by the European Union payed for my rent.

Thanks to these institutions.

Special thanks to my supervisor, Martin Hellwig, for his guidance and patience.

Contents

Chapter 1

Introduction

”*n Spieler spielen ein gegebenes Gesellschaftsspiel. Wie muß einer dieser Spieler spielen, um dabei ein möglichst günstiges Resultat zu erzielen?*

Die Fragestellung ist allgemein bekannt, und es gibt wohl kaum eine Frage des täglichen Lebens, in die dieses Problem nicht hineinspielte; trotzdem ist der Sinn dieser Frage kein eindeutig klarer. Denn sobald $n > 1$ ist (d.h. ein eigentliches Spiel vorliegt), hängt das Schicksal eines Spielers außer von seinen eigenen Handlungen auch noch von denen seiner Mitspieler ab; und deren Benehmen ist von genau denselben egoistischen Motiven beherrscht, die wir beim ersten Spieler bestimmen möchten. Man fühlt, daß ein gewisser Zirkel im Wesen der Sache liegt.”

John von Neumann (1928)

In his essay ”Zur Theorie der Gesellschaftspiele” John von Neumann has characterized the central problem of game theory as follows: How does an individual decide if the evaluation of his decision depends on the decisions made by other individuals, for whom the evaluation of their decision depends in turn on their opponents' decisions? In the terminology of game theory such a situation of strategic interdependency is a 'game' and the individuals are 'players' who have to choose between different 'strategies' and who evaluate the 'outcome' resulting from all players' strategy choices. In order to answer the question 'Which strategies will be chosen in a game?' game theory has developed solution concepts for games that can be roughly divided in two different approaches: the equilibrium approach and the rationalizability approach.

The equilibrium approach presumes that the players' strategy choices are an equilibrium point in the sense of Nash (1950b), i.e., each player's strategy is a best response against the strategies of his opponents. Suppose each player would form expectations about the play of his opponents and he would choose his optimal strategy given these expectations. We can then interpret a Nash equilibrium as a strategy profile such that each player's expectation is confirmed by the actual strategy choices of his opponents. Because no player has to revise his expectations concerning his opponents' strategy choices in an equilibrium it appears as plausible that players end up in equilibrium when a game is repeated over and over. Thus, as a justification for equilibrium play we can imagine some 'learning' mechanism, running in the background, which leads to a stable coordination of strategy choices satisfying the definition of an equilibrium point. However, for games that are just 'one-shot' strategic situations or that are not often repeated this 'coordination via learning'-argument in favor of equilibrium points is not available. For these games the question of how players shall arrive at correct expectations about opponents' strategy choices may be difficult to answer.

The rationalizability approach tries to solve a game by eliminating 'unreasonable' strategies. The starting point of the rationalizability approach is the assumption that a player will not choose strategies which are not best responses against any strategy profile of his opponents. If this assumption results in an elimination of strategies the complexity of the problem is reduced. In a next step the rationalizability approach assumes that a player will not choose strategies which are not best responses against any remaining strategy profiles of his opponents in the reduced problem; and so on... Thus, by iteration we may arrive at some set of 'rationalizable' strategies that can not be further reduced.

Rationalizability presumes now that players engage in this process of reasoning such that only rationalizable strategies will be chosen in the course of a game. Unlike the equilibrium approach the rationalizability approach does not require players to have correct expectations about opponents' strategy choices, and it claims only that a player expects his opponents to play some rationalizable strategy. Moreover, the process of reasoning, as assumed by the rationalizability approach, does not presuppose any learning from previous play such that we can expect players to choose rationalizable strategies even in 'one-shot' strategic situations; at least when the players are indeed strategically sophisticated enough to go through the necessary iterations.

Let me explain both approaches by a simple example. The following payoff-matrix depicts each player's evaluation of the possible strategy-profiles of a game (in normal form) I would like to call "Education" game

	go on	*stop*
encourage	2,2	0,0
ignore	1,1	1,0
discourage	0,0	0,1

The equilibrium approach predicts for the Education game that player A 'encourages' and that player B 'goes on': If player A expects player B to 'go on' he chooses to 'encourage' because this strategy gives him with 2 the highest payoff among his possible strategy choices. Accordingly, player B would 'go on' if he expects player A to 'encourage' such that the strategy profile 'encourage, go on' is an equilibrium point. Moreover, by inspecting the remaining five strategy-profiles of this game we see that there does not exist any other equilibrium point.

The rationalizability approach also predicts that player A will 'encourage' and player B will 'go on'. However, the rationalizability approach applies a quiet different reasoning for arriving at this unique 'rationalizable' solution of the game. If we look at A's strategy choices we can conclude that he will never 'discourage' because whatever B is doing 'discourage' is not a best response. Let A also realize this, and let him furthermore assume that B realizes this too, i.e., B knows that A will not 'discourage'. Having eliminated 'discourage' as a strategy that will not be chosen by A, and that will not be regarded by B as a possible choice of A, we are left with a strategic situation given by

	go on	*stop*
encourage	2,2	0,0
ignore	1,1	1,0

Rationalizability claims that A should be aware of the fact that B will not choose 'stop' because it is not a best response for B in this new strategic situation. As a consequence A considers only the reduced strategic situation

	go on
encourage	2,2
ignore	1,1

Here the strategic situation boils down to a simple decision-problem for A. He will choose 'encourage' as his unique best response, and we have arrived at a unique rationalizable strategy-choice for A. If B follows such a reasoning as well he will choose 'go on', and we obtain as unique rationalizable strategy profile of the Education game 'encourage, go on'.

*

In the Education game the equilibrium approach and the rationalizability approach arrive at the same unique solution, but in general rationalizability concepts are by construction weaker solution concepts than equilibrium concepts. Even if there is a unique equilibrium point we may encounter many rationalizable strategies. The following payoff matrix of a game, called "Adventure World", depicts such a situation where every individual strategy is rationalizable whereas 'float, fight' is the unique equilibrium point (in pure strategies).

	run	$fight$	$hide$
$rise$	0,1	0,0	1,0
$float$	0,0	1,1	0,0
$sink$	1,0	0,0	0,1

When we consider games for which the assumption of correct expectations is hardly justified the weakness of the rationalizability approach can bear interpretational advantages over the equilibrium approach: the equilibrium approach may rule out too many 'reasonable' strategies that might actually be chosen by the players. However, this conceptual weakness can become a problem for the usefulness of the rationalizability approach as a positive theory: If only a few strategies are excluded as 'unreasonable' the rationalizability approach has not much predictive power.

From the point-of-view of predictive power rationalizability should ideally identify a unique rationalizable strategy profile, as in the Education game. Moreover, for games with a unique rationalizable solution (and only for these games) the rationalizability approach gains the status of a normative theory that can recommend to all players the 'right' strategy choice: If we advised each player to go through all the stages of reasoning such that each player arrives at a unique strategy the resulting play will be a Nash equilibrium, i.e., no player has an incentive to deviate from his rationalizable strategy given the other players stick to their rationalizable strategies as well. Thus, for games with a unique rationalizable strategy profile we can offer a good answer to the equilibrium approach's problem of how players could arrive in 'one-shot' games at correct expectations about opponents' strategy choices: The players have just to engage in the internal process of reasoning, as presumed by the rationalizability approach, and they will end up with correct expectations about their opponents' strategy choices.

In chapter 2 of this thesis I am going to characterize classes of games with a unique rationalizable solution. As the central mathematical condition of my uniqueness results I introduce the concept of T-contractivity. Contractivity, (or 1-contractivity in my terminology), of the best-response function states that for arbitrary strategy profiles the 'distance' between the best-response strategy profiles against these strategy profiles has to be smaller than the distance between these strategy profiles. T-contractivity postulates the same property for a T-fold application of the best response function. Existing uniqueness results in the literature refer to 1-contractivity (Moulin 1984) or even stronger contraction-properties (Bernheim 1984) of the best response function. Moreover, these existing results are restricted to very specific strategy sets (closed intervals of the real line).

One main result in chapter 2 shows that T-contractivity (respectively the stronger condition of T-contraction) in any mathematical distance is a sufficient condition for a unique solution in arbitrary compact (respectively complete and bounded) sets of strategy profiles. For games with finite strategy sets T-contractivity is even a necessary uniqueness condition. Furthermore, a slightly weaker condition than T-contractivity turns out to be a sufficient and necessary for a unique rationalizable solutions in games with monotonic best response functions (including so-called 'supermodular games', see Topkis 1979; Vives 1990; Milgrom and Roberts 1990). Finally, because a unique rationalizable solution must be a unique equilibrium point, the uniqueness results of chapter 2 are also relevant to the equilibrium approach: Games with unique Nash equilibria can be identified; technics become available for the computation of these equilibrium points.

For real-valued strategy sets and differentiable best-response functions sufficient conditions for T-contractivity can be stated in terms of the first order derivatives of the best-response functions. In chapter 3 of this thesis I apply these uniqueness conditions for differentiable best response functions to a problem originally proposed by Bernheim (1984) and extended by Basu (1992): In standard Cournot-Oligopolies with sufficiently many firms every output-decision between zero and the monopoly-output becomes rationalizable **even** if there is a unique Nash equilibrium! I show that for any given number of firms suitable assumptions concerning the market-impact of firms and their cost functions imply a unique rationalizable output-decision for each firm. For instance, we can always guarantee unique rationalizable strategies in Cournot-Oligopolies in case the products of the firms are sufficiently heterogenous. Because real-life firms rarely compete on markets of perfectly homogenous goods the uniqueness results of chapter 2 would endow us with good predictions about market-outcomes whenever Cournot Competition appears as the relevant model. This predictive power holds even for 'one-shot' strategic situations of Cournot Competition for which the equilibrium approach has less appeal than the rationalizability approach.

*

Even more important than uniqueness is the question of existence for a solution concept. If there does not exist an equilibrium-point, or if there does not exist a rationalizable strategy profile, then neither a prediction about nor a recommendation for a player's strategy choice is possible. Due to Nash's (1950a; 1950b) existence result we take the existence of equilibrium points (in mixed strategies) as granted for games with finite strategy sets. Moreover, because every equilibrium is rationalizable the existence of rationalizable strategies is implied by Nash's proof as well. But Nash had derived his existence result under the specific assumption of Expected Utility maximizing players; an assumption which attracted recently strong criticism due to its lack of realistic appeal.

Von Neumann and Morgenstern (1947) had introduced the idea that the strategy choice of a player can be described as a choice between 'lotteries' (=probability distributions with finite support). A best response of a player, given his expectations about his opponents' strategy choices, is then a preference-maximizing choice among some set of lotteries. Von Neumann and Morgenstern presumed that a player's preferences over lotteries satisfy some axioms such that these preferences become representable by an Expected Utility functional: a player's evaluation of a lottery is computed as a sum of utility-numbers, assigned to deterministic outcomes, which are weighted with the probabilities of the outcomes.

To assume Expected Utility maximizing players is technically very convenient, however, real individuals systematically violate EU-maximizing behavior (see, e.g., Allais 1953; Kahneman and Tversky 1979). Even for idealized 'rational' individuals it is not necessarily obvious why they should stick to the Expected Utility axioms. As a reaction to this criticism of EU-theory several decision-theoretic models have been developed with the aim to avoid the flaws of the Expected Utility assumptions (for an overview see, e.g., Karni and Schmeidler 1991; Schmidt 1998; Starmer 2000).

For these alternative (Non-Expected Utility) models of decision-making the existence of equilibrium-points may break down. Nevertheless, even if Nash equilibria do not exist, there exist so-called 'equilibria in beliefs' (Crawford 1990) as long as the players' preferences are representable by continuous utility functions. Equilibria in beliefs are a re-interpretation of Nash equilibria in terms of correctly held beliefs instead of actually chosen strategies. In particular, for a game with n players an equilibrium in beliefs is defined as a n-tupel of beliefs such that i.) each player expects with positive chance only strategies that are best responses of his opponents against their beliefs, and ii.) any two players share the same belief concerning the strategy-choice of any third player. Thus, for Non-EU models with continuously representable preferences we obtain existence of equilibrium-points that are re-defined as equilibria in beliefs.

The assumption of a continuous utility representation is crucial for this existence result, however, there are psychologically motivated decision-theoretic models, so-called 'security- and potential level preferences' (Gilboa 1988; Jaffray 1988; Cohen 1992; Essid 1997) which violate this assumption. In chapter 4 of this thesis I investigate the existence of strategic solutions for players with security- and potential level preferences. The results of chapter 4 establish that existence of equilibria in beliefs may fail whereas existence of rationalizable strategies is guaranteed. The fact that it can be impossible for two preference-maximizing players to have mutually correct expectations concerning their strategy choices casts in my opinion a severe doubt on the claim that solutions of strategic situations must be equilibrium points.

The relevance of the existence vs. non-existence results of chapter 4 depends on the relevance of the security and potential level preference models; especially, on their assumption on the occurrence of discontinuous preferences. Consider a "Pre-emptive Strike" game where player A can either wag a 'war' or keep 'peace' while a player B will 'destroy' or 'distribute' weapons of mass destruction. Let player A have the following payoff-matrix

	destroy	*distribute*
peace	2	0
war	1	1

Suppose now that A 'does not take any chances': If A was sure that B 'destroys' his WMD he chooses 'peace'; in contrast, A chooses 'war' whenever he believes that B will 'distribute' WMD with positive chance.

Such choice behavior of A is in my opinion perfectly reasonable as well as relevant, however, the according preferences are not continuously representable: if the probability of B 'distributing' WMD drops to zero there would be a discontinuous upward-jump in A's welfare by keeping 'peace'. Due to this discontinuity the existence result for equilibria in beliefs (Crawford 1990) does not apply here anylonger. SL,PL-preferences can take account of this upward-jump, and by the notion of a 'security level effect' they offer even a convincing psychological explanation for this discontinuity in A's evaluation of lotteries (compare Lopes 1987).

Besides the possibility of 'upward jumps' SL,PL-preferences have other advantages which add to their realistic appeal. They allow, for example, also for 'downward jumps' in the evaluation of lotteries in case risky choices lose any chance on good outcomes. But most importantly SL,PL-preferences can explain (via these discontinuities) so-called Allais paradoxa (Allais 1953) which are the most prominent violations of EU-theory. The motivation for SL,PL-preferences, in particular their approach to Allais paradoxa, will be further discussed in chapter 4 of this thesis.

*

There exist different rationalizability concepts in the literature which refer basically to the same process of internal reasoning that a player presumably uses when he wants to determine a 'reasonable' strategy choice for himself:

1. "All players choose only a best response for some belief."
2. "My opponents know stage 1."

...

i. "All players will only choose a best response for some belief that is consistent with the knowledge at stage i-1."

i+1. "My opponents know i."

and so forth...

Furthermore, all rationalizability concepts formalize a player's beliefs as probability-distributions over his opponents' strategy-choices such that a 'belief is consistent with the knowledge at all stages' if and only if opponents' strategies, excluded at some stage, do not appear in the support of the belief. However, these concepts differ by their assumptions which probability-distributions shall be regarded as admissible beliefs at the starting point of the reasoning process.

The concept of 'point-rationalizability' considers only degenerated probability-one distributions ('point beliefs'). 'Independent rationalizability', usually just called 'rationalizability', restricts possible beliefs to probability distributions that assume independently chosen strategies of the opponents. Finally, the concept of 'correlated rationalizability' allows for arbitrary probability-distributions such that a player may believe his opponents can correlate their strategy choices in any possible way. Correlated rationalizability is in general weaker than independent rationalizability which is in turn weaker than point-rationalizability, i.e., each point-rationalizable strategy is also a rationalizable strategy whereas the converse is not necessarily true.

In chapter 5 of this thesis I investigate conditions for which these different rationalizability concepts determine the same set of rationalizable strategies. Such equivalence conditions are of interest because they characterize games for which the interpretational advantages of independent rationalizability and of correlated rationalizability are combined with the technical simplicity of point-rationalizability. Moreover, for games satisfying these equivalence conditions, the question becomes irrelevant whether the assumption of arbitrary beliefs or of beliefs restricted to independent strategy choices is more appropriate for a given situation.

To see the interpretational shortcomings of point-rationalizability consider the following payoffs of a player A in the "Circus Dompteur" game

	beat	*bite*
allow	2	-1Mill
assist	1	1
avoid	-1Mill	2

To 'assist' is not a best choice against any point-belief of A: if A believes that B 'beats' then A would 'allow', and if A believes that B 'bites' then A would 'avoid'. However, suppose A conceives 'beat' and 'bite' as equally likely. Then it may be reasonable for A to 'assist' because he could avoid by this choice the 0.5 chance of losing 1Mill.

The reason for working with point-rationalizability at all, and not with other rationalizability concepts instead, is its technical convenience: It is much easier to determine the best responses against pure strategies (point-beliefs) than to determine the best responses against all probability mixtures over these pure strategies. For example, the uniqueness conditions of chapter 2 are at first derived for point-rationalizable strategies because I can apply mathematical tools to point-rationalizability that are not at hand for the other rationalizability concepts (contraction properties of the best response function against pure strategies). Only in a second step these uniqueness results are extended to correlated rationalizability by an application of equivalence results obtained in chapter 5. As another consequence point-rationalizability is the most prominent rationalizability concept applied in economical models. But this often leaves the question open whether we could have obtained the same solution of the model for more convincing rationalizability concepts, or not.

The equivalence conditions of chapter 5 characterize now games for which the technical convenience of point-rationalizability goes along with the conceptual advantage of the other rationalizability concepts. Central conditions for the equivalence of all rationalizability concepts refer to properties of the utility functions (quasiconcave utility functions; supermodular utility functions with monotonic differences) and they refer to properties of the strategy sets (compact intervals of the real line; complete lattices).

The findings of chapter 5 will restate and extend existing equivalence results (e.g., Milgrom and Roberts 1990 for supermodular games with a unique point-rationalizable strategy). For example, the equivalence results of chapter 5 apply also to models of Cournot-Oligopolies which are not supermodular in the sense of Milgrom and Roberts.

Chapter 5 investigates also equivalence conditions for rationalizability concepts and for the closely related solution concepts of so-called 'iterated elimination of dominated strategies'. There exist famous results in the literature that relate both families of solution concepts (Pearce 1984; Moulin 1984; Milgrom and Roberts 1990) and I will provide restatements as well as extensions of these results. Interestingly, all the equivalence of chapter 5 do not require the standard assumption of Expected Utility maximizing players: simple stochastic dominance conditions, implied by monotonicity with respect to first order stochastic dominance, are already sufficient.

* * *

Chapter 2

Uniqueness Of Rationalizable Solutions

2.1 Introduction

This chapter explores sufficient and necessary conditions for the uniqueness of rationalizable strategies with the aim to characterize classes of normal form games with a unique rationalizable solution. Rationalizability concepts (Bernheim 1984; Pearce 1984) try to solve a game by the elimination of 'unreasonable' strategies. In a first step it is assumed that every player of a game chooses only a best response against some belief about his opponents' strategy choices. In a second step all players shall be aware of this situation such that every player chooses only best responses against some belief about his opponents' remaining strategy choices. By iteration of this argument we may finally arrive at a set of 'rationalizable' strategies which can not be further reduced.

Another approach to solving games is the equilibrium approach. Equilibrium concepts presume that the solution of a game is a Nash equilibrium (Nash 1950b), i.e., a strategy profile such that the strategy of each player is a best response against the strategies of his opponents. If players choose a best response against some belief a Nash equilibrium requires the players to have correct beliefs about their opponents' strategy choices. In contrast when a player chooses a rationalizable strategy as a best response against some belief this belief is not necessarily correct. As a consequence rationalizability concepts are weaker solution concepts for normal form games than equilibrium concepts: every Nash equilibrium is a rationalizable strategy profile whereas the converse is not necessarily true.

To explain why players should have correct beliefs may be quite difficult for equilibrium concepts; especially for 'one-shot' strategic situations. Rationalizability concepts do not have this problem, however, they face other difficulties. First, a justification of rationalizable strategies as solutions requires a degree of 'strategic sophistication' on behalf of all players which is often unrealistic. Second, the conceptual weakness of rationalizability concepts can become a problem for the usefulness of these solution concepts: if there are too many rationalizable strategies there is not much predictive power.

This first difficulty is not avoided for games with a unique rationalizable solution. However, under the assumption of strategically sophisticated players the rationalizability approach gains for these games maximal predictive power. Moreover, games with a unique rationalizable solution are also of interest for the equilibrium approach because rationalizability offers then a good explanation why players may arrive at correct beliefs even in 'one-shot' games. Finally, for games with a unique rationalizable solution rationalizability can gain the status of a normative theory: Suppose we recommend all players to go through the strategic reasoning of the rationalizability approach. If we can convince each player that his opponents will do the same no player will deviate from this recommendation.

In Moulin (1984) and in Bernheim (1984) sufficient conditions for a unique rationalizable strategy are presented that are based on contraction properties of the best response function. Moulin shows that contraction of the best response function, i.e., the distance between any two best-responses against strategy profiles is smaller than the distance between these strategy profiles, is a sufficient condition for uniqueness. This sufficient condition for uniqueness is more general than Bernheim's condition who requires a sufficiently fast contraction such that the speed of the contraction has to increase in the number of players. Both, Moulin and Bernheim, apply very specific definitions of the 'mathematical distance' and they restrict their results to games with individual strategy sets that are subsets of the real numbers (respectively of R^n).

Milgrom and Roberts (1990) show that the rationalizable solution of a so-called 'supermodular' game (compare also Topkis 1979; Vives 1990) is unique if and only if the Nash equilibrium of this game is unique. In contrast to the results of Bernheim and of Moulin, Milgrom and Roberts' uniqueness result is applicable to games with quite general strategy sets (as long as these strategy sets are 'complete lattices'). On the other hand the assumption of 'supermodular' games is rather restrictive; for example, all individual best response functions must be increasing. Moreover, the problem of a sufficient and necessary condition for a unique rationalizable strategy in supermodular games is merely translated into the problem of a sufficient and necessary condition for a unique Nash equilibrium whereby Milgrom and Roberts do not provide such a uniqueness condition.

This chapter relaxes Bernheim's and Moulin's uniqueness conditions with respect to contraction properties of the best response function as well as with respect to possible strategy sets. As central mathematical condition I introduce the concept of T-contractivity, i.e., the distance between strategy profiles that result from a T-fold application of the best response function to starting-point strategies is smaller than the distance between these starting-point strategies. One main result of this chapter shows that T-contractivity, defined in an arbitrary distance, is a sufficient condition for uniqueness. This result is valid for arbitrary strategy sets that are compact subsets of some metric space. (For strategy sets that are only complete and bounded subsets of a metric space the slightly stronger condition of T-contraction is needed to guarantee uniqueness.) For games with finite strategy sets T-contractivity turns is also a necessary uniqueness condition.

A slightly weaker condition than T-contractivity turns out to be a sufficient and necessary uniqueness condition for all games with monotonic best response functions. This includes the class of supermodular games but it applies also to games with decreasing best response functions. As a consequence Milgrom and Roberts' uniqueness result for supermodular games can be described as a special case of the contraction approach. Moreover, as a side result, we can characterize now supermodular games with a unique Nash equilibrium by this sufficient and necessary condition for a unique rationalizable solution.

The uniqueness results of this chapter will be at first derived for the so-called concept of point-rationalizability. Compared to independent, or to correlated, rationalizability, the concept of point-rationalizability has severe interpretational flaws because only best responses against probability-one beliefs are considered. In general a unique point-rationalizable strategy is not necessarily the unique correlated rationalizable strategy. However, in chapter 5 of this thesis several conditions are identified which guarantee that a unique point-rationalizable strategy is also the unique correlated rationalizable strategy of a game. Whenever possible these conditions are applied in this chapter.

The remainder of this chapter is organized as follows. Section 2 introduces basic definitions. In section 3 the main uniqueness results for metrizable strategy sets are presented. In section 4 even stronger results are obtained under the additional assumptions of monotonic best response functions and of strategy sets that are complete lattices. Section 5 presents derived results for differentiable best response functions and individual strategy sets that are subsets of the reals. Section 6 provides a brief outlook on possible economical applications. All proofs are relegated to the appendix.

2.2 Definitions

For a given set of players I let $G = (S_i, f_i)_{i \in I}$ denote a game in normal form with S_i as individual strategy set of player i. By $f_i : \triangle(S_{-i}) \to 2^{S_i}$ I denote the individual best response correspondence such that f_i maximizes player i's preference ordering over the set $S_i \times \triangle(S_{-i})$ with $\triangle(S_{-i})$ as set of probability distributions over $S_{-i} = \times_{j \neq i} S_j$. An element $\sigma_{-i} \in \triangle(S_{-i})$ will be called a 'belief' of player i about the strategy choices of his opponents. In case σ_{-i} is a 'point-belief', i.e., assigns probability one to some strategy profile s_{-i}, I simply write $f_i(s_{-i})$ instead of $f_i(\sigma_{-i})$. For single-valued $f_i(\sigma_{-i})$ I write $s_i = f_i(\sigma_{-i})$ instead of $s_i \in f_i(\sigma_{-i})$. If all individual best response correspondences against point-beliefs are single-valued I call $f: S \to S$, with $f(s) = \times_{i=1}^{I} f_i(s_{-i})$, a 'best response I-function'.

Definition. *(*Bernheim 1984; Pearce 1984*)*

The set of point-rationalizable strategies of a game G is given by

$$P(G) = \bigcap_{k=0}^{\infty} \lambda^k(S)$$

,such that $\lambda^k(S) = \times_{i=1}^{I} \lambda_i^k(S)$ *with*

$$\lambda_i^k(S) = \bigcup_{s_{-i} \in \lambda_{-i}^{k-1}(S)} f_i(s_{-i})$$

and $\lambda_{-i}^0(S) = S_{-i}$.

The set of correlated rationalizable strategies of a game G is given by

$$R^C(G) = \bigcap_{k=0}^{\infty} \mu^k(S)$$

,such that $\mu^k(S) = \times_{i=1}^{I} \mu_i^k(S)$ *with*

$$\mu_i^k(S) = \bigcup_{\sigma_{-i} \in \triangle\left(\mu_{-i}^{k-1}(S)\right)} f_i(\sigma_{-i})$$

and $\mu_{-i}^0(S) = S_{-i}$.

Notice, that for games with a best-response I-function the set of point-rationalizable strategies is equivalently given by $P(G) = \bigcap_{k=0}^{\infty} \lambda^k(S)$ with $\lambda^k(S) = \bigcup_{s \in \lambda^{k-1}(S)} f(s)$ and $\lambda^0(S) = S$. For some number $T \in N$ let the function $f^T: S \to S$ be inductively defined by $f^T(s) = f\left(f^{T-1}(s)\right)$ with $f^0(s) = s$.

Definition. *Given some* $T \in N$.

The best response I-function f *is said to be T-contractive if* f^T *is contractive, i.e.,* $d\left(f^T(s), f^T(t)\right) < d(s,t)$ *for all* $s,t \in S$ *with* $s \neq t$.

The best response I-function f *is said to be a T-contraction if* f^T *is a contraction, i.e., there exists some* $c \in (0,1)$ *such that* $d\left(f^T(s), f^T(t)\right) \leq c * d(s,t)$ *for all* $s,t \in S$.

2.3 Uniqueness Results For Metric Spaces

2.3.1 Results

In this section strategy sets are considered that can be described as subsets of some arbitrary metric space. The results of proposition 1 and of proposition 2 are valid for an arbitrary distance function d.

Proposition 1. *Given a game* G *such that*

(A1) The best response I-function f *exists and is continuous.*

(A2) S *is a compact, non-empty subset of some metric space* (X,d).

Then there exists a unique point-rationalizable strategy of G *if the best response I-function* f *is T-contractive.*

A point-rationalizable strategy of G *is unique only if there is some* $c \in (0,1)$ *such that there exists for all* $s,t \in S$ *some* $T \in N$*, dependent on* s,t*, with* $d\left(f^T(s), f^T(t)\right) \leq c * d(s,t)$.

Given the assumptions (A1) and (A2) of proposition 1 T-contractivity in some distance for some $T \geq 1$ is a sufficient condition for a unique point-rationalizable strategy. In contrast, the necessary condition of proposition 1 does not require the same number T such that $d\left(f^T(s), f^T(t)\right) \leq c * d(s,t)$ for all $s, t \in S$. The following example shows that T-contractivity is not a necessary condition for uniqueness:

Example. $I = 2$, $S_i = [0,1]$, and $f_1(s_2) = (0.5)^{s_2}$ and $f_2(s_1) = 1 - s_1$. This game satisfies the assumptions of proposition 1 and it can be shown that there exists a unique point-rationalizable strategy given by $s^* = (1,0)$, (f is non-increasing and we can apply condition i.) of proposition 3 since $\lim_{k\to\infty} \left\|f^k(s) - f^k(t)\right\|_1 = 0$ for $s = (0,0)$ and $t = (1,1)$). On the other hand T-contractivity in $\|\cdot\|_1$ is violated: Suppose $s(M) = (s_1 = 1, s_2 = 0)$ and $t(M) = \left(t_1 = 1 - \frac{1}{M}, t_2 = 0\right)$ and observe that for any fixed T the number M can be chosen sufficiently large to obtain $\left\|f^T(s(M)) - f^T(t(M))\right\|_1 \geq \|s(M) - t(M)\|_1$.

It remains an open question whether the necessary condition of proposition 1 is also a sufficient condition for uniqueness under the assumptions (A1) and (A2). However, we can apply proposition 1 to derive a sufficient + necessary condition for finite games.

Consider the case of a game G with finite strategy set S such that $\#S = m > 0$. Let us endow S with the discrete topology induced by the discrete metric d, such that $d(s,t) = 1$ if $s \neq t$, and $d(s,t) = 0$ if $s = t$. Observe that the assumptions (A1), (A2) of proposition 1 are satisfied in case $f_i(s_{-i})$ is single-valued for all $s_{-i} \in S_{-i}$ and all i. Hence, there exists a unique point-rationalizable solution of G if f is a T-contraction for some finite T. Furthermore, in the appendix it is shown that there must exist some T not greater than m if there exists any T at all such that $d\left(f^T(s), f^T(t)\right) < d(s,t)$ for any $s, t \in S$.

As a simple consequence we obtain the following corollary which implies that uniqueness of a point-rationalizable strategy for a game G with finite strategy set S, with $\#S = m$, and single-valued best response correspondences f can be determined by a finite number of computations, (with $m * m$ as upper bound).

Corollary 1. *Given a finite game G with $\#S = m$ and with best-response I-function. There exists a unique point-rationalizable strategy of G if and only if $f^m(s) = f^m(t)$ for all $s, t \in S$.*

The next proposition provides a sufficient condition under a relaxation of the compactness-assumption in proposition 1.

Proposition 2. *Given a game G such that*

(A1) *The best response I-function f exists and is continuous.*

(A2) *S is a nonempty, bounded and complete subset of a metric space (X, d).*

Then there exists a unique point-rationalizable strategy of G if the best response I-function f is a T-contraction.

Non-emptiness of $P(G)$ in proposition 2 is guaranteed by a fixed-point theorem (basically an extension of the famous Banach fixed-point theorem from 1-contractions to T-contractions). This was not necessary for proposition 1. Instead, the proof of proposition 1 establishes as a side-result the existence of a unique fixed-point for a T-contractive continuous single-valued mapping from a compact set into itself, (compactness is necessary if T-contractivity is assumed instead of T-contraction, compare the fixed-point theorem for 1-contractivity in Bonsall 1962).

If S_i is a subset of R^n proposition 2 obtains as a special case of proposition 1 because completeness and boundedness of S_i imply then compactness of S_i. However, proposition 2 is useful for more general strategy-spaces for which compactness might fail (or is not easily assured) whereas completeness and boundedness are satisfied. Consider the example of a symmetric two-player game G with S_i as the set of functions $b_i : [0,1] \to [0,1]$, and with individual best response function $f_i(b_j) = 0.9 * b_j$. Take the metric space (S_i, d) with d as the "uniform metric" (induced by the supremums-norm), and notice that S is a nonempty, bounded, and complete subset of the product space $(S_1, d) \times (S_2, d)$ with $d(s,t) = \max\{d(s_1,t_1), d(s_2,t_2)\}$, (compare Theorem 3 p.92 in Berge 1997). It is easy to see that the best response I-function $f(b) = (f_i(b_j), f_j(b_i))$ is a 1-contraction since

$$d\left(f\left(b^1\right), f\left(b^2\right)\right) \leq 0.9 * d\left(b^1, b^2\right)$$

for all $b^1, b^2 \in S$. This proves the existence of a unique point-rationalizable strategy of G by proposition 2, whereas proposition 1 was not applicable because S is not compact.

2.3.2 Relation to the Uniqueness Results of Bernheim and of Moulin

In Bernheim (1984) and in Moulin (1984) appear already two sufficiency conditions for unique point-rationalizable strategies that refer to contraction-mapping properties of the best response I-function.

Bernheim (1984) offers by proposition 5.5 a sufficiency condition for uniqueness of point-rationalizable strategies in terms of the best response I-function which states basically: A point-rationalizable strategy is unique if the best response I-function is a contraction mapping in the Euclidean metric with a sufficiently fast contraction whereas the speed of the contraction has to increase in the numbers of players in a game. In particular, the contraction speed is characterized by the following formula with $d(s,t) = \|s-t\|_2$, $s,t \in S$, and S as compact subset of R^n

$$d(f(s), f(t)) < \frac{d(s,t)}{\sqrt{(I-1)}}$$

A look into Moulin's (1984) proof of Theorem 4 reveals that Moulin had been well aware that contraction of the best-response I-function in the supremums-norm implies a unique point-rationalizable strategy if the S_i are compact subsets of R. Moulin uses this uniqueness condition for point-rationalizable strategies to derive a uniqueness condition for weak dominance solution against pure strategies in so-called 'nice' games, (compare the remarks in section 5 of this chapter). Notice that point-rationalizable and correlated rationalizable strategies coincide for a class of games which implies Moulin's nice' games, (for details see chapter 5 of this thesis). Thus, applied to this class of games proposition 1 states a sufficient and a necessary condition for the uniqueness of a correlated rationalizable strategy.

Obviously, the results of this section are more general than the results of Bernheim and of Moulin: they are applicable to strategy sets that are subsets of an arbitrary metric space, and they consider T-contractivity, resp. T-contraction, for $T \geq 1$. But even under the restrictions that $T = 1$ and that the strategy spaces are compact subsets of R^n the sufficiency condition of proposition 1 offers useful extensions to the sufficiency conditions in

Bernheim (1984) and Moulin (1984): Proposition 1 relaxes 1-contraction (or "sufficiently fast" contraction as in Bernheim) to 1-contractivity. Furthermore, proposition 1 shows that 1-contraction under **any** metric is a sufficient condition for uniqueness, (note: 1-contraction in a specific norm does not necessarily imply 1-contraction in another norm).

2.4 Lattice-Structures And Monotonic Best Response Functions

2.4.1 Results

In this section strategy sets are considered that can be simultaneously described as a subset of a metric space (X, d) **and** as a lattice $(S, \leq_L)$. Useful uniqueness results obtain then for monotonic best response I-functions.

Recall at first some notions of lattice theory (compare Topkis 1979; Vives 1990; Milgrom and Roberts 1990; Fudenberg and Tirole 1996). Given a reflexive, transitive, and antisymmetric binary relation $\leq_L$ on a set S_i let $(S_i, \leq_L)$ denote a lattice, i.e., for all elements $s_i, t_i \in S_i$ there exist a supremum $s_i \vee t_i$ and an infimum $s_i \wedge t_i$ in S_i. Furthermore, $(S, \leq_L)$ denotes then a lattice given by the product-order, i.e., $s \leq_L t$ iff $s_i \leq_L t_i$ for all i. A lattice $(S, \leq_L)$ is complete if $\inf T \in S$ and $\sup T \in S$ for every non-empty subset $T \subset S$. In particular, completeness of $(S, \leq_L)$ implies the existence of exactly one "smallest" element $s \in S$ such that $s \leq_L s'$ for all $s' \in S$, and of exactly one "largest" element $t \in S$ such that $s' \leq_L t$ for all $s' \in S$. Order-continuity of f on a complete lattice $(S, \leq_L)$ implies for every chain C (=totally ordered subset of S) $\lim_{s\in C, s\downarrow \inf C} f(s) = f(\inf C)$ and $\lim_{s\in C, s\uparrow \sup C} f(s) = f(\sup C)$.

Proposition 3. *Given a game G such that*

(A1) S is a bounded subset of some metric space (X, d) and a complete lattice $(S, \leq_L)$ such that $d(s', t') \leq d(s, t)$ whenever $s \leq_L s', t'$ and $s', t' \leq_L t$.

(A2) The best response I-function exists and is order-continuous.

Furthermore, either one of the following two conditions is satisfied:

(A3a) The best response I-function is non-decreasing: if $s \leq_L t$ then $f(s) \leq_L f(t)$,

or

(A3b) The best response I-function is non-increasing: if $s \leq_L t$ then $f(t) \leq_L f(s)$

.

Then there exists a unique point-rationalizable strategy of G if and only if one of the following conditions is satisfied

i.) $\lim_{k\to\infty} d\left(f^k(s), f^k(t)\right) = 0$ with s as the smallest and with t as the largest element in S, or

ii.) There exists for all $s, t \in S$, with $s \neq t$, some $T \in N$, dependent on s, t, such that $d\left(f^T(s), f^T(t)\right) < d(s, t)$.

Under the assumptions of proposition 3 the necessary condition of proposition 1 becomes a necessary and a sufficient condition for unique point-rationalizable strategy. Assumption (A1) claims a particular relationship for the partial order $\leq_L$ and the distance d: the distance between the "smallest" and the "largest" element of some lattice should not be smaller than the distance between an arbitrary pair of elements in this lattice.

Recall that a normed Riesz space is an ordered vector space which is a lattice as well as a metric space with norm-induced metric, (compare the chapters 6-7 in Aliprantis and Border 1994). Consequently, whenever all S_i are normed Riesz spaces the strategy set S can be characterized as a lattice and as a subset of a metric space under the max-norm $\|s\| = \max_{i\in I} \|s_i\|$. Typical individual strategy sets of economic interest should be describable as normed Riesz spaces such that assumption (A1) of proposition 3 is satisfied for S. To see this consider the following three examples.

Example. Let S_i be a subset of the Riesz space $B(X)$ of all bounded real functions on X under the supremums-norm $\|s_i\|_\infty = \sup\{|s_i(x)| \mid x \in X\}$. Let us impose the following lattice structure on S_i: $s_i \leq_L t_i$ if and only if $s_i(x) \leq t_i(x)$ for all $x \in X$. Suppose now that $s_i \leq_L s_i', t_i'$ and $s_i', t_i' \leq_L t_i$, and without restricting generality assume further that $\|t_i' - s_i'\|_\infty = \sup\{t_i'(x) - s_i'(x) \mid x \in X\}$. Since $t_i(x) \geq t_i'(x)$ and $s_i'(x) \geq s_i(x)$ for all $x \in X$ we have $\|t_i - s_i\|_\infty \geq \|t_i' - s_i'\|_\infty$.

Example. Let S_i be a subset of the Riesz space l_∞ of all continuous real functions on N with compact support, i.e.,

$$l_\infty = \left\{s_i \in R^N \mid \|s_i\|_\infty < \infty\right\}$$

This example is obviously a special case of the first example under the assumption $X = N$. Since l_∞ is nothing else than the space of sequences with bounded entries we can conclude that for typical settings of dynamic games with infinite time-horizon the individual strategy sets S_i can be described as a lattice and as a subset of a metric space such that assumption (A1) of proposition 3 is satisfied, (compare the "Arms race"-example of a supermodular game in Milgrom and Roberts 1990).

Example. Let S_i be a subset of the Riesz space $B([0,1])$ of all bounded real functions on $[0,1]$ under the L_1-norm, i.e. $\|s_i\| = \int_0^1 |s_i(x)|\,dx$, such that s_i and t_i are considered as identical iff $\int_0^1 |s_i(x) - t_i(x)|\,dx = 0$. Let us impose the following lattice structure on S_i: $s_i \leq_L t_i$ if and only if the set $\{x \mid s_i(x) > t_i(x)\}$ is of measure zero, (compare Theorem 3 in Milgrom and Roberts, 1990). Suppose now $s_i \leq_L s_i', t_i'$ and $s_i', t_i' \leq_L t_i$. Notice that

$$\begin{aligned} \int_0^1 t_i(x) - s_i'(x)\,dx + \int_0^1 t_i(x) - t_i'(x)\,dx &= d(s_i', t_i) + d(t_i, t_i') \geq d(s_i', t_i') \\ \int_0^1 t_i'(x) - s_i(x)\,dx + \int_0^1 s_i'(x) - s_i(x)\,dx &= d(t_i', s_i) + d(s_i, s_i') \geq d(t_i', s_i') \end{aligned}$$

Summing up the l.h.s and the r.h.s of the above inequalities gives the desired result

$$\begin{aligned} 2\int_0^1 t_i(x) - s_i(x)\,dx &\geq d(s_i', t_i') + d(t_i', s_i') \\ d(s_i, t_i) &\geq d(s_i', t_i') \end{aligned}$$

Two distinct notions of 'completeness' are applied in this chapter: the uniqueness condition of proposition 2 assumes a complete subset S of a metric space in the sense that every Cauchy-sequence in S converges, whereas the uniqueness results of this section assume that S is a (order-) complete lattice. Norm complete Riesz spaces are called Banach lattices but the reader should be aware that such norm-complete Riesz-spaces are not necessarily (order-) complete lattices.

For example, it is well-known that the set of continuous real functions on the unit-interval $C[0,1]$ is not a complete lattice, (compare Aliprantis and Border 1994). Furthermore, if $C[0,1]$ is equipped with the L_1-norm it is not a Banach-lattice either while it is a normed Riesz-space, (compare Vassiliev 2001). However, $C[0,1]$ is a complete metric space, i.e., a Banach lattice, if it is equipped with the supremums-norm. As a consequence, it is not possible to establish uniqueness of point-rationalizable solutions by an application of the results derived in this section to games with individual strategy sets given by $C[0,1]$. Furthermore, we can not obtain uniqueness-results by proposition 2 for such games if $C[0,1]$ is equipped with the L_1-norm. Nevertheless, proposition 2 is immediately applicable if we choose instead the supremums-norm.

Thus, despite the usefulness of the lattice-approach, there might exist relevant cases for which uniqueness-results can not be established via the properties of the lattice-structure of S but via the properties of S as a subset of an appropriate metric space.

2.4.2 Relation to the Uniqueness Result of Milgrom and Roberts

Milgrom and Roberts (1990) show that the correlated rationalizable strategy of a supermodular game G is unique if and only if the Nash equilibrium of G is unique. Recall that the best response I-function of a supermodular game is non-decreasing, and observe that the arguments in the proof of proposition 3 are immediately applicable in order to obtain the following relations of Nash equilibria and of point-rationalizable solutions in games with monotonic best response I-functions.

Corollary 2: *Suppose a game G satisfies the assumptions (A1) and (A2) of proposition 3.*

If the best response I-function is non-decreasing, i.e., (A3a), then there exists a "smallest" element $s \in P(G)$ and a "largest" element $t \in P(G)$ such that $f(s) = s$ and

$f(t) = t$, i.e., s and t are Nash equilibria. Consequently, there exists a unique point-rationalizable strategy of G if and only if the Nash equilibrium of G is unique.

If the best response I-function is non-increasing, i.e., (A3b), then there exists a "smallest" element $s \in P(G)$ and a "largest" element $t \in P(G)$ such that $f(s) = t$, $f^2(s) = s$ and $f(t) = s$, $f^2(t) = t$, i.e., s and t are Nash equilibria if and only if $P(G)$ is single-valued.

Under the additional assumptions of supermodular utility-functions and of monotonic utility differences the uniqueness result of proposition 3 can be extended from point-rationalizable to correlated rationalizable strategies. Suppose that player i's preference ordering over the set $S_i \times \triangle(S_{-i})$ satisfies monotonicity with respect to first order stochastic dominance and that it is represented by some utility function $U_i : S_i \times \triangle(S_{-i}) \rightarrow R$ such that $f_i(\sigma_{-i}) = \arg\max_{s_i \in S_i} U_i(s_i, \sigma_{-i})$. The utility function U_i is supermodular on S_i if for all $s_i, t_i \in S_i$

$$U_i(s_i, s_{-i}) + U_i(t_i, s_{-i}) \leq U_i(s_i \wedge t_i, s_{-i}) + U_i(s_i \vee t_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$. Furthermore, U_i has "increasing differences" if $U_i(s_i, s_{-i}) - U_i(t_i, s_{-i})$ is non-decreasing in s_{-i} for $t_i \leq_L s_i$, and it has "decreasing differences" if $U_i(s_i, s_{-i}) - U_i(t_i, s_{-i})$ is non-increasing in s_{-i} for $t_i \leq_L s_i$.

Theorem (chapter 5 of this thesis): *Given a game G that satisfies the assumptions (A1) and (A2) of proposition 3 as well as the following assumptions*

(A3) Each U_i is supermodular on S_i.

(A4) Each U_i has either increasing or decreasing utility differences.

Then the unique point-rationalizable strategy of G is also the unique correlated rationalizable strategy of G.

For a supermodular game i.) each U_i is supermodular on S_i and has increasing differences, and ii.) S is a complete lattice. Furthermore, each U_i satisfies order-uppersemicontinuity in s_i and order-continuity in s_{-i}; an assumption that is actually encompassed in proposition 3 by the claim for order-continuity of f whenever a best response I-function exists. Since increasing utility differences imply a non-decreasing best response I-function the corollary, together with the theorem, restates Milgrom and Roberts' uniqueness result for supermodular games with a best response I-function.

By proposition 3 we can conclude that a correlated rationalizable strategy of games with monotonic best response I-functions, satisfying the assumptions of the theorem, is unique if and only if there exists for all $s, t \in S$ with $s \neq t$ some $T \in N$, dependent on s, t, such that $d\left(f^T(s), f^T(t)\right) < d(s, t)$. Thus, in addition to Milgrom and Roberts' result we have obtained a sufficient and necessary condition for unique correlated rationalizable strategies even if the best response I-function is non-increasing.

In the case of non-decreasing best response I-functions this condition is also necessary and sufficient for a unique Nash equilibrium. This is not anylonger true for games with non-increasing best response I-functions: T-contractivity is then not a necessary condition for a unique Nash equilibrium. For example, Bernheim (1984) and Basu (1990) observe for a standard model of Cournot oligopoly that every individual output between zero and the monopoly-output becomes point-rationalizable for games with more than two firms. Such a Cournot oligopoly has a unique Nash equilibrium and it satisfies the assumptions of proposition 3 whereas the response I-function is decreasing.

The following example shows that monotonic best response functions establish only uniqueness of the point-rationalizable strategies whereas the stronger condition of monotonic utility differences are needed to guarantee a unique correlated rationalizable strategy. Suppose the utility numbers of EU-maximizers are given by

	$b1$	$b2$
$a1$	$2, 0$	$2, 3$
$a2$	$0, 2$	$3, 0$
$a3$	$3, 3$	$0, 2$

Let $a1 \leq_L a2 \leq_L a3$ and observe that f is non-increasing if $b1 \leq_L b2$, (respectively non-decreasing if $b2 \leq_L b1$). There exists a unique point-rationalizable strategy by $(a3, b1)$, i.e., there exists some finite T such that f^T is contraction with d as the discrete metric. However, there exist many correlated rationalizable strategies since $a1$ is a best response against the belief that $b1$ and $b2$ are equally likely, and so forth. Supermodularity of each U_i in S_i is trivially satisfied but U_A does not have decreasing (respectively increasing) differences.

2.5 Differentiable Best Response Functions

For real valued and continuously differentiable individual best response functions T-contraction of f is guaranteed if the partial derivatives satisfy specific conditions. Recall the definition of the function f^T and notice that a partial derivative evaluated at s, $\frac{\partial f_i^T}{\partial s_j}(s)$, is computed via successive applications of the chain-rule:

$$\begin{aligned}
\frac{\partial f_i^1}{\partial s_j}(s) &= \frac{\partial f_i}{\partial s_j}(s) \\
\frac{\partial f_i^T}{\partial s_j}(s) &= \sum_{k \neq i} \frac{\partial f_i}{\partial s_k} \frac{\partial f_k^{T-1}}{\partial s_j}(s)
\end{aligned}$$

Proposition 4: *Given a game G such that*

(A1) Each S_i is a non-empty, compact, and convex subset of R.

(A2) Each individual best response function f_i is continuously differentiable.

Then there exists a unique point-rationalizable strategy of G

i.) *if for each player i*

$$\sum_{j \in I} \left| \frac{\partial f_i^T}{\partial s_j}(s) \right| < 1 \tag{2.1}$$

for all $s \in S$, for some $T \geq 1$, **or**

ii.) if for each player j

$$\sum_{i \in I} \left| \frac{\partial f_i^T}{\partial s_j}(s) \right| < 1 \tag{2.2}$$

for all $s \in S$, for some $T \geq 1$.

The results of proposition 4 can be considered as special cases of the Mean-Value Inequality for functions $R^n \to R^n$ (compare Heuser 1998), i.e.,

$$\left\| f^T(s) - f^T(t) \right\| \leq \left| Df^T(r) \right|_M * \|s - t\|$$

such that $Df^T(r) = \left(\frac{\partial f_i^T}{\partial s_j}(r)\right)_{i=1,\ldots,I;j=1,\ldots I}$ denotes the matrix of first-order partial derivatives and $r \in S$ maximizes some matrix-norm $|\cdot|_M$ over the elements in

$$\{\lambda s + (1-\lambda)\, t \mid \lambda \in [0,1]\}$$

which is compatible with the norm $\|\cdot\|$. For example, condition (2.1) implies T-contraction of f in the supremums-norm $\|\cdot\|_\infty$ because the "absolute row-sum"-norm of a matrix is compatible with the supremums-norm $\|\cdot\|_\infty$ (see Heuser1998, p57f.). Analogously, the condition (2.2) can be derived because the "absolute column-sum"-norm of a matrix is compatible with the "absolute value"-norm $\|\cdot\|_1$. Consequently, whenever there is for some norm $\|\cdot\|$, with $S \subset (X,d)$ and $d(s,t) = \|s-t\|$, a compatible matrix-norm $|\cdot|_M$ such that $\left|Df^T(r)\right|_M < 1$ for some $T \geq 1$, then the existence of a unique point-rationalizable solution is proved. As a further example of "compatible matrix-norms" notice that the "spectrum"-norm of a matrix, given by

$$\left|Df^T(r)\right|_M = \max Eigenvalue\left(Df^T(r) * \left(Df^T(r)\right)^{trans}\right)$$

is compatible with the Euclidean norm $\|\cdot\|_2$. Hence, whenever the maximal singular value of $Df^T(r)$ is strictly smaller than 1 for all $r \in S$ the point-rationalizable solution must be unique.

The conditions of proposition 4 are most easily checked for 1-contraction. For $T = 1$ condition (2.1) turns then into

$$\sum_{j \neq i} \left|\frac{\partial f_i}{\partial s_j}(s)\right| < 1$$

for all i and all $s \in S$, and condition (2.2) becomes

$$\sum_{i \neq j} \left|\frac{\partial f_i}{\partial s_j}(s)\right| < 1$$

for all j and all $s \in S$. The first of these two contraction-conditions is already implied by theorem 4 in Moulin (1984): in the proof of theorem 4 Moulin shows that an equivalent formulation of this contraction-condition, (in terms of second-order partial derivatives of the utility-functions), implies a unique point-rationalizable strategy.

Proposition 5: *Given a game G such that*

(A1) Each S_i is a non-empty, compact, and convex subset of R.

(A2) Each individual best response function f_i is continuously differentiable.

Then there does **not** *exist a unique point-rationalizable strategy of G*

i.) if there exist for each $T \geq 1$ two strategy-profiles $s, t \in S$, with $s \neq t$, such that for all $j \in I$

$$\sum_{i \in I} \left| \frac{\partial f_i^T}{\partial s_j}(r) \right| \geq 1$$

for $r \in \{\lambda s + (1-\lambda)\, t \mid \lambda \in [0,1]\}$, **or**

ii.) if there exist for each $T \geq 1$ two strategy-profiles $s, t \in S$, with $s \neq t$ and $s_k = s_l$, $t_k = t_l$, for all $k, l \in I$, such that for all $i \in I$

$$\sum_{j \in I} \left| \frac{\partial f_i^T}{\partial s_j}(r) \right| \geq 1$$

for $r \in \{\lambda s + (1-\lambda)\, t \mid \lambda \in [0,1]\}$.

To see how the conditions of proposition 4 and 5 work in practice consider the following example that can be derived from a particular model of Cournot competition between three firms. Let $S_i = [0,1]$ for $i \in \{1,2,3\}$ and suppose the individual best response functions are given by

$$\begin{aligned} f_1(s_{-1}) &= -0.5s_2 - 0.5s_3 \\ f_2(s_{-2}) &= -0.5s_1 - 0.5s_3 \\ f_3(s_{-3}) &= -0.5s_1 - (0.5-\varepsilon)\, s_2 \end{aligned}$$

for some $\varepsilon \in [0, 0.5]$. Consider the case $\varepsilon = 0$. It can be shown by induction that for all $T \geq 1$

$$\sum_{j \in I} \left| \frac{\partial f_i^T}{\partial s_j}(s) \right| = 1$$

for all $i \in I$ and all $s \in S$. Hence, by proposition 5 there does not exist a unique point-rationalizable strategy if $\varepsilon = 0$.

Consider now the case $\varepsilon > 0$. Obviously, f is not a 1-contraction by the conditions (2.1) or (2.2) of proposition 4. However, it is a 2-contraction in the supremums-norm as well as in the absolute value-norm, e.g., condition (2.1) is satisfied because $\sum_{j \in I} \left| \frac{\partial f_i^2}{\partial s_j}(s) \right| \leq 1 - \frac{\varepsilon}{2}$ for all i and all $s \in S$:

$$\begin{aligned} f_1^2(s) &= 0.5s_1 + \left(0.25 - \frac{\varepsilon}{2}\right) s_2 + 0.25s_3 \\ f_2^2(s) &= 0.25s_1 + \left(0.5 - \frac{\varepsilon}{2}\right) s_2 + 0.25s_3 \\ f_3^2(s) &= \left(0.25 - \frac{\varepsilon}{2}\right) s_1 + 0.25s_2 + \left(0.5 - \frac{\varepsilon}{2}\right) s_3 \end{aligned}$$

Consequently, there exists a unique point-rationalizable strategy whenever $\varepsilon > 0$.

2.6 An Outlook On Possible Applications

For a typical model of a sealed-bid auction Battigalli and Siniscalchi (2000) show that every non-zero bid below the equilibrium is rationalizable. Dekel and Wolinsky (2001) can obtain a unique rationalizable solution for sealed-bid auctions with very specific model-parameters (e.g., many players, finite action sets). The uniqueness results of this chapter are applicable to games with rather general strategy sets such as Bayesian games with continuous type space that are typically used for a formalization of sealed-bid auctions. Thus, by the results of this chapter we can try to identify, or to design, models of auctions with a unique rationalizable solution.

For example, in reality we encounter sealed-bid auctions for which the winner is not exclusively determined by the highest bid but additionally by side-considerations of the auctioneer which leave leeway for speculations by the bidders, e.g., credibility of bidders (Kirch-Media and Erste Fußball Bundesliga), or the overall-business concept (Camelot vs. The People's Lottery). If behavioral assumptions like 'optimistic' or 'pessimistic' bidders (with respect to the auctioneer's side-considerations) are formalized as properties of the best response functions I would expect that we can establish T-contraction and therefore unique solutions.

For a standard model of Cournot oligopoly Bernheim (1984) and Basu (1990) observe that every output between zero and the monopoly output becomes rationalizable if there are sufficiently many firms in the oligopoly. This result is a severe blow to the attractiveness of rationalizability as a solution concept; especially, because the Nash equilibrium is typically unique for these models.

By an application of the uniqueness conditions for differentiable best response functions conditions for any number of firms are derived in chapter 3 of this thesis which imply unique rationalizable output decisions in a Cournot oligopoly. One key assumption refers to 'not-perfectly homogenous goods': if firms compete with eachother on sufficiently separated product-markets the rationalizable solution will be unique. Because this assumption has some realistic appeal rationalizability may offer good predictions for models of Cournot oligopolies with strategically sophisticated players.

2.7 Appendix: Proofs

Proof of proposition 1:

The "if"-part. Observe at first that each set $\lambda^k(S)$, $k \geq 0$, is compact and non-empty because continuity of $f(s)$ inherits compactness and non-emptiness. Since $\lambda^k(S) \subset \lambda^{k-1}(S)$, for $k \geq 1$, $P(G) = \bigcap_{k=0}^{\infty} \lambda^k(S)$ is compact and non-empty as an infinite intersection of compact and non-empty nested sets, (compare the proof of proposition 3.1 in Bernheim 1984). In order to prove single-valuedness of $P(G)$ it is sufficient to show that $\lim_{k\to\infty} diam\left(\lambda^k(S)\right) = 0$, with

$$diam\left(\lambda^k(S)\right) = \sup\left\{d(s,t) : s,t \in \lambda^k(S)\right\}$$

By compactness of $\lambda^k(S)$ and continuity of $d : X \times X \to R^+$ there must exist $s^k, t^k \in \lambda^k(S)$ such that $diam\left(\lambda^k(S)\right) = d\left(s^k, t^k\right)$. Observe that $d\left(s^k, t^k\right)_{k\geq 1}$ is a monotonically decreasing numerical sequence, bounded from below, that converges to it's lower bound. Since $S \times S$ is compact there exists a converging subsequence $\lim_{k'\to\infty}\left(s^{k'}, t^{k'}\right) = (s,t)$ such that $\lim_{k\to\infty} diam\left(\lambda^k(S)\right) = d(s,t)$. Because $f^M(s)$ is continuous for any $M \geq 1$ there exist for every set $\lambda^{k+M}(S)$, with fixed $M \geq 1$, some elements $f^M\left(s^k(M)\right), f^M\left(t^k(M)\right) \in \lambda^{k+M}(S)$ such that $diam\left(\lambda^{k+M}(S)\right) = d\left(f^M\left(s^k(M)\right), f^M\left(t^k(M)\right)\right)$ and $s^k(M), t^k(M) \in$

$\lambda^k(S)$. Since by assumption $d\left(s^k(M), t^k(M)\right) \le d\left(s^k, t^k\right)$ for $k \ge 1$ there is some subsubsequence with $\lim_{k'' \to \infty}\left(s^{k''}, t^{k''}\right) = (s,t)$ such that $\lim_{k'' \to \infty}\left(s^{k''}(M), t^{k''}(M)\right) = (s(M), t(M))$ and $d(s(M), t(M)) \le d(s,t)$. Furthermore, by continuity of f^M

$$\lim_{k'' \to \infty}\left(f^M\left(s^{k''}(M)\right), f^M\left(t^{k''}(M)\right)\right) = \left(f^M(s(M)), f^M(t(M))\right)$$

with $\lim_{k \to \infty} diam\left(\lambda^k(S)\right) = d\left(f^M(s(M)), f^M(t(M))\right)$ for $M \ge 1$. Suppose now that there exists for all pairs $s(M) \ne t(M)$, $M \ge 1$, the same $T \in N$ such that

$$d\left(f^T(s(M)), f^T(t(M))\right) < d(s(M), t(M))$$

Choosing $M = T$ gives

$$d\left(f^M(s(M)), f^M(t(M))\right) < d(s(M), t(M))$$

whenever $s(M) \ne t(M)$. Since $d\left(f^M(s(M)), f^M(t(M))\right) < d(s,t)$ we obtain the contradiction $\lim_{k \to \infty} diam\left(\lambda^k(S)\right) \ne \lim_{k \to \infty} diam\left(\lambda^k(S)\right)$ whenever $\lim_{k \to \infty} diam\left(\lambda^k(S)\right) \ne 0$.

The "only if"-part. Given some s, t by compactness and non-emptiness of $S \times S$ there exists some converging subsequence such that $\lim_{T' \to \infty}\left(f^{T'}(s), f^{T'}(t)\right) = (s^*, t^*)$. By continuity of d we obtain $\lim_{T' \to \infty} d\left(f^{T'}(s), f^{T'}(t)\right) = d(s^*, t^*)$. Due to $s^*, t^* \in P(G)$ a unique point-rationalizable solution requires $d(s^*, t^*) = c(s,t) * d(s,t)$ with $c(s,t) = 0$ if $s \ne t$, or with some arbitrary number $c(s,t)$ if $s = t$. Take some ε, with $0 < \varepsilon < 1$, and observe that by continuity of d and of $f^{T'}$ there must exist for all $(s,t) \in S \times S$ some finite T, dependent on (s,t), such that $d\left(f^T(s), f^T(t)\right) \le \varepsilon * d(s,t)$. $\square$

Proof of corollary 1: If $f^m(s) = f^m(t)$ for all $s, t \in S$ then $d(f^m(s), f^m(t)) < d(s,t)$ for all $s \neq t$ since $d(f^m(s), f^m(t)) = 0$. Hence, f is m-contractive and there exists a unique point-rationalizable solution by proposition 1. It remains to show that m-contractivity is also necessary. Consider the finite sequence $s, f(s), f^2(s), ..., f^m(s)$ of $m+1$ elements and observe that there must occur some $f^k(s)$ and $f^h(s)$, with $k < h$, such that $f^k(s) = f^h(s)$. If $f^{m-1}(s) \neq f^m(s)$ then there can not exist a unique point-rationalizable solution since $f^k(s), f^{k+1}(s), ..., f^{h-1}(s), f^h(s) \in P(G)$. Consequently, a unique point-rationalizable solution requires $f^{m-1}(s) = f^m(s)$ and $f^{m-1}(t) = f^m(t)$ for any $s, t \in S$. Suppose now there exists a unique point-rationalizable solution and assume $f^m(s) \neq f^m(t)$ for some $s, t \in S$. By $f^{m-1}(s) = f^m(s)$ and $f^{m-1}(t) = f^m(t)$ there does not exist any finite T such that $d\left(f^T(f^m(s)), f^T(f^m(t))\right) < d(f^m(s), f^m(t))$, violating the necessary condition of proposition 1.□

Proof of proposition 2: If S is complete then continuity and T-contraction of the best response I-function imply the existence of a (unique) fixed-point (compare Theorem 1.3 in Bonsall 1962). This guarantees $P(G) \neq \emptyset$. Furthermore, since S is bounded, and $\lambda^k(S) \subset \lambda^{k-1}(S)$, there exists for each set $\lambda^k(S)$ a finite diameter $diam\left(\lambda^k(S)\right)$which converges to its lower bound. Since $P(G)$ is non-empty $P(G)$ is single-valued if $\lim_{k\to\infty} diam\left(\lambda^k(S)\right) = 0$, (compare the proof of Theorem 3.10 b in Rubin 1976). Due to $diam\left(\lambda^k(S)\right) \geq d\left(s^k, t^k\right)$ for all $s^k, t^k \in \lambda^k(S)$ T-contraction implies $d\left(s^{k+T}, t^{k+T}\right) \leq c * diam\left(\lambda^k(S)\right)$ for all $s^{k+T}, t^{k+T} \in \lambda^{k+T}(S)$. Observe that this implies in turn $diam\left(\lambda^{k+T}(S)\right) \leq c{*}diam\left(\lambda^k(S)\right)$ for $k \geq 0$: if there is some $s^{k+T}, t^{k+T} \in \lambda^{k+T}(S)$ with $d\left(s^{k+T}, t^{k+T}\right) = c * diam\left(\lambda^k(S)\right)$

then $diam\left(\lambda^{k+T}(S)\right) = c * diam\left(\lambda^{k}(S)\right)$, and if $d\left(s^{k+T}, t^{k+T}\right) < c * diam\left(\lambda^{k}(S)\right)$ for all $s^{k+T}, t^{k+T} \in \lambda^{k+T}(S)$ then $diam\left(\lambda^{k+T}(S)\right) \leq c * diam\left(\lambda^{k}(S)\right)$. Consequently

$$\lim_{m\to\infty} diam\left(\lambda^{m*T}(S)\right) \leq \lim_{m\to\infty} c^{m} * diam(S) = 0$$

for $c < 1$.□

Proof of proposition 3: Condition i.) will become obvious by the proof of condition ii.) where it is shown that $\lim_{k\to\infty} d\left(f^{k}(s), f^{k}(t)\right) = diam(P(G))$ with s as smallest and t as largest element in S.

Ad condition ii.) **The "if"-part.** By lattice-completeness of S and monotonicity of f there exist for all $k \geq 1$ strategies $s^{k}, t^{k} \in \lambda^{k}(S)$ such that $s^{k} \leq_{L} s'$ and $s' \leq_{L} t^{k}$ for all $s' \in \lambda^{k}(S)$. Hence, by assumption (A1) $diam\left(\lambda^{k}(S)\right) = d\left(s^{k}, t^{k}\right)$. Suppose condition (A3a) is satisfied. By assumption (A3a) $f\left(s^{k}\right) \leq_{L} f(s'), f(t')$ and $f(s'), f(t') \leq_{L} f\left(t^{k}\right)$ for all $s', t' \in \lambda^{k}(S)$, and by (A1) we obtain $diam\left(\lambda^{k+1}(S)\right) = d\left(f\left(s^{k}\right), f\left(t^{k}\right)\right)$. A repeated application of this argument gives $diam\left(\lambda^{k+T}(S)\right) = d\left(f^{T}\left(s^{k}\right), f^{T}\left(t^{k}\right)\right)$ for any $T \geq 1$. Observe that $\left(t^{k}\right)_{k\geq 1}$ is a monotonically decreasing sequence, bounded from below, and $\left(s^{k}\right)_{k\geq 1}$ is a monotonically increasing sequence, bounded from above. Because S is complete the order-limits $t^{*} = \inf t^{k}$ and $s^{*} = \sup s^{k}$ exist such that $diam(P(G)) = d(s^{*}, t^{*})$. Analogously, by order-continuity of f^{T} it is also true that $diam(P(G)) = d\left(f^{T}(s^{*}), f^{T}(t^{*})\right)$ for any $T \geq 1$. Consequently, $P(G)$ must be unique if there exists for any pair $s \neq t$ a finite T such that $d\left(f^{T}(s), f^{T}(t)\right) < d(s, t)$.

Suppose now that condition (A3b) is satisfied. We obtain for $s^{k} \leq_{L} s', t'$ and $s', t' \leq_{L} t^{k}$ that $f\left(t^{k}\right) \leq_{L} f(s'), f(t')$ and $f(s'), f(t') \leq_{L} f^{k}(s)$. Hence, by assumption

(A1) $d\left(f\left(s'\right), f\left(t'\right)\right) \leq d\left(f\left(t^k\right), f\left(s^k\right)\right)$, for all $s', t' \in \lambda^k(S)$. By definition $d\left(f\left(t^k\right), f\left(s^k\right)\right) = d\left(f\left(s^k\right), f\left(t^k\right)\right)$ which implies again $d\left(f\left(s^k\right), f\left(t^k\right)\right) = diam\left(\lambda^{k+1}(S)\right)$. Analogously, we obtain $diam\left(\lambda^{k+T}(S)\right) = d\left(f^T\left(s^k\right), f^T\left(t^k\right)\right)$ for any $T \geq 1$. Observe now that the sequence $f\left(r^k\right)_{k\geq 0}$ with $r^k = t^k$, for $k = 0, 2, 4, \ldots$, and $r^k = s^k$, for $k = 1, 3, 5, \ldots$, is monotonically increasing, and bounded from above, whereas the sequence $f\left(q^k\right)_{k\geq 0}$ with $q^k = t^k$, for $k = 1, 3, 5, \ldots$, and $q^k = s^k$, for $k = 0, 2, 4, \ldots$, is monotonically decreasing, and bounded from below. By completeness of S and order-continuity of f^T it follows $diam(P(G)) = d(f(r), f(q))$ as well as $diam(P(G)) = d\left(f^T(f(r)), f^T(f(q))\right)$ for any $T \geq 1$ and we obtain the desired result.

The "only if"-part. Suppose condition (A3a) is satisfied. Since $diam(P(G)) = d(s^*, t^*)$ we have $diam(P(G)) = 0$ only if $\lim_{k\to\infty} d\left(s^k, t^k\right) = 0$. But if there exist some $s', t' \in S$ with $s' \neq t'$ and $d\left(f^T(s'), f^T(t')\right) \geq d(s', t')$ for all T then $\lim_{k\to\infty} d\left(s^k, t^k\right) \geq d(s', t') > 0$ since $d\left(s^k, t^k\right) \geq d\left(f^k(s'), f^k(t')\right)$ for all k. Analogously for (A3b).$\Box$

Proof of proposition 4:

Part i.). Let $g_i(\lambda) = f_i^T(\lambda(s-t)+t)$, and observe that $g_i(\lambda)$ is continuously differentiable on $[0,1]$. The Mean-Value Inequality for real-valued functions with a real-valued domain implies then

$$|g_i(1) - g_i(0)| \leq \left|\frac{\partial g_i}{\partial \lambda}(\lambda^*)\right| * |1-0| \tag{2.3}$$

for some $\lambda^* = \arg\max_{[0,1]} \left|\frac{\partial g_i}{\partial \lambda}(\lambda)\right|$, which must exist. By an application of the chain-rule:

$$\frac{\partial g_i}{\partial \lambda}(\lambda^*) = \sum_{j\in I} \frac{\partial f_i^T}{\partial s_j}(\lambda^*(s_j - t_j) + t_j) * (s_j - t_j)$$

$$\left|\frac{\partial g_i}{\partial \lambda}(\lambda^*)\right| \leq \left|\sum_{j\in I}\frac{\partial f_i^T}{\partial s_j}(\lambda^*(s_j - t_j) + t_j)\right| * \|s-t\|_\infty$$

According substitution for the terms in the inequality (2.3) gives

$$\left|f_i^T(s) - f_i^T(t)\right| \leq \left|\sum_{j\in I}\frac{\partial f_i^T}{\partial s_j}(r)\right| * \|s-t\|_\infty$$

with $r = \lambda^*(s-t) + t$. Since this is true by assumption for all $i \in I$ we obtain for the supremums-norm

$$\left\|f^T(s) - f^T(t)\right\|_\infty \leq \left|\sum_{j\in I}\frac{\partial f_i^T}{\partial s_j}(r)\right| * \|s-t\|_\infty$$

Consequently, the assumption $\sum_{j\in I}\left|\frac{\partial f_i^T}{\partial s_j}(s)\right| < 1$ for all i and all $s \in S$ guarantees T-contraction of f in the supremums-norm. (Notice: T-contraction, and not only T-contractivity, derives from the fact that $\sum_{j\in I}\left|\frac{\partial f_i^T}{\partial s_j}(s)\right|$ is a continuous function that obtains a maximum on the compact set S. Consequently, if $\sum_{j\in I}\left|\frac{\partial f_i^T}{\partial s_j}(s)\right| < 1$ for all i and all $s \in S$ then there must exist some $c < 1$ such that $\sum_{j\in I}\left|\frac{\partial f_i^T}{\partial s_j}(s)\right| \leq c$ for all i.)

Part ii.) is accordingly proved by showing that $\sum_{i\in I}\left|\frac{\partial f_i^T}{\partial s_j}(s)\right| < 1$ for all j and all $s \in S$ implies T-contraction of f in the "absolute value"-norm, (compare the proof of part i.) of proposition 5 below).□

Proof of proposition 5:

Part i.) Let again $g_i(\lambda) = f_i^T(\lambda(s-t)+t)$, and observe that the Mean-Value Inequality implies

$$|g_i(1) - g_i(0)| \geq \left|\frac{\partial g_i}{\partial \lambda}(\lambda^i)\right| * |1-0|$$

for some $\lambda^i = \operatorname{argmin}_{[0,1]}\left|\frac{\partial g_i}{\partial \lambda}(\lambda)\right|$. By an according application of the chain-rule and substitution

$$\left|f_i^T(s) - f_i^T(t)\right| \geq \left|\sum_{j\in I} \frac{\partial f_i^T}{\partial s_j}\left(r^i\right) * (s_j - t_j)\right| \tag{2.4}$$

with $r^i = \lambda^i s + \left(1 - \lambda^i\right) t$. Summing up over all i and rearranging

$$\begin{aligned}
\sum_{i\in I} \left|f_i^T(s) - f_i^T(t)\right| &\geq \sum_{i\in I} \left|\sum_{j\in I} \left(\frac{\partial f_i^T}{\partial s_j}\left(r^i\right)\right) * (s_j - t_j)\right| \\
\sum_{i\in I} \left|f_i^T(s) - f_i^T(t)\right| &\geq \min_{j\in I} \left\{\left|\sum_{i\in I} \left(\frac{\partial f_i^T}{\partial s_j}\left(r^i\right)\right)\right|\right\} * \sum_{j\in I} |s_j - t_j| \\
\left\|f^T(s) - f^T(t)\right\|_1 &\geq \|s - t\|_1
\end{aligned}$$

Where the last step follows from the assumption $\sum_{i\in I}\left(\frac{\partial f_i^T}{\partial s_j}\left(r^i\right)\right) \geq 1$ for all j and all $r^i \in \{\lambda s + (1-\lambda) t \mid \lambda \in [0,1]\}$. Consequently, $d\left(f^T(s), f^T(t)\right) \geq d(s,t)$ with d induced by the "absolute value"-norm. By the necessary condition of proposition 1 there does not exist a unique point-rationalizable solution if this is true for all $T \geq 1$.

Part ii.) Proceeding as in part i.) we obtain inequality (2.4). By assumption $(s_j - t_j) = (s_k - t_k)$ for all $j, k \in I$ such that $\|s - t\|_\infty = |s_j - t_j|$ for $j \in I$ implies

$$\left|f_i^T(s) - f_i^T(t)\right| \geq \left|\sum_{j\in I} \frac{\partial f_i^T}{\partial s_j}\left(r^i\right)\right| * \|s - t\|_\infty$$

And because $\left|\sum_{j\in I} \frac{\partial f_i^T}{\partial s_j}\left(r^i\right)\right| \geq 1$ is true for all i we obtain $d\left(f^T(s), f^T(t)\right) \geq d(s,t)$ with d induced by the supremums-norm.$\square$

Chapter 3

Cournot Oligopolies With Unique Rationalizable Solutions

3.1 Introduction

Rationalizability concepts (Bernheim 1984; Pearce 1984) try to solve a game by the exclusion of 'unreasonable' strategies. Consider a player who goes through the following internal monologue:

1. "Every player will only choose some best response."

2. "My opponents know this."

...

i. "Every player will only choose a best response given the knowledge at stage i-1."

i+1. "My opponents know i."

and so forth...

At each stage of this reasoning-process the player may identify 'unreasonable' strategies that are not best responses given the considerations he has made so far. When the player arrives at some stage such that no further strategies are excluded he is left with his 'rationalizable' strategies. Rationalizability concepts presume now that the players will only choose some rationalizable strategy in the course of a game.

A different approach to the solution of a game are equilibrium concepts. Equilibrium concepts presume that players will play a Nash equilibrium, i.e., each player chooses a best response against the strategies which are chosen by his opponents. If we consider players who form expectations about their opponents' strategy choices a Nash equilibrium implies that all players have had correct expectations. It can be difficult for equilibrium concepts to explain how players are able to make these correct anticipations; especially, for strategic situations that do not occur frequently with the same participants. For such strategic situations, e.g., 'one-shot' games, rationalizability looks then like the more appropriate solution concept. (At least when we can presume players who are 'strategically sophisticated' enough for involving in the internal monologue...)

Rationalizability concepts can encounter the problem that there are too many rationalizable strategies such that rationalizability becomes useless as a solution concept because it has no predictive power at all. In contrast, equilibrium concepts are stronger than rationalizability concepts such that there are games which possess many rationalizable solutions whereas they have a unique Nash equilibrium.

For example, Bernheim (1984) shows for a standard model of Cournot oligopoly with a unique Nash equilibrium that any output between zero and the Monopoly-output is rationalizable if there are more than two firms in the oligopoly. Basu (1992) obtains the same multitude of rationalizable strategies for a general class of Cournot oligopolies if the number of firms is sufficiently high. Cournot oligopolies are the standard models in industrial organization theory for output competition among firms. It is therefore quite disappointing that rationalizability concepts, with their interpretational advantages, do rather badly as solution concepts for these important games.

This chapter explores conditions for which Cournot oligopolies have a unique correlated rationalizable strategy such that the rationalizability approach has ultimate predictive power for Cournot oligopolies satisfying these conditions. The findings of this chapter suggest that the situation is not as bad as it appears by the results of Bernheim and Basu: Uniqueness can already be obtained under small parameter changes in Bernheim's original model of Cournot oligopoly.

In particular, I investigate the impact of two different model-specifications. First, I relax Bernheim's assumption that the products of all firms are perfect substitutes, i.e., homogenous goods. Instead I introduce the notion of a 'negative externality' Cournot oligopoly where firms have basically their own product-markets but may face different degrees of competition from the output decisions of other firms. Secondly, I deviate from the assumption of linear cost-functions. Small changes in the 'homogenous good' assumption as well as in the cost-function parameters can be sufficient for uniqueness. Moreover, for any number of firms uniqueness can be obtained if the impact of the other firms output on a firm's own market is sufficiently small.

The above reasoning process is typically the weakest assumption we impose in models of 'strategically rational' players (compare Guesnerie 2002). The existence of a unique rationalizable strategy looks then as a very 'robust' solution whenever the firms of the oligopoly can be considered as 'strategically rational' or 'strategically sophisticated' players. Moreover, for Cournot oligopolies with a unique rationalizable strategy the equilibrium approach can offer a good explanation for the question how firms arrive at correct expectations about their competitors: each firms goes simply through the process of reasoning as stated above. As a consequence the equilibrium solution of these Cournot oligopolies is a convincing solution even if we consider Cournot competition restricted to one-shot strategic situations.

3.2 Uniqueness Conditions

A game $G = (S_i, U_i)_{i \in I}$ in normal form will be called a Cournot oligopoly if $S_i = [0,1]$ and $U_i(s) = P_i(s)\, s_i - C_i(s_i)$, with $P_i : S \rightarrow R^+$ and $C_i : S_i \rightarrow R^+$. The function P_i is interpreted as the inverse demand function, i.e., for a given market output it determines the price pro unit output firm i can take for its product. C_i denotes the cost function, and S_i is the set of possible output-decisions of firm i.

Let $f_i : \triangle(S_{-i}) \rightarrow 2^{S_i}$ denote best response correspondence of firm i which maximizes i's preference ordering over the set $S_i \times \triangle(S_{-i})$. The set $\triangle(S_{-i})$ denotes all probability distributions over $S_{-i} = \times_{j \neq i} S_j$, and I will interpret an element of $\triangle(S_{-i})$ as firm i's 'belief' over the output decisions of its competitors. For the results of this chapter the firms are not required to be Expected Utility maximizers. However, it will be assumed that firm i's preferences over $S_i \times \triangle(S_{-i})$ satisfy monotonicity with respect to first order stochastic dominance.

The set of correlated rationalizable strategies (Bernheim 1984; Pearce 1984) of G is defined as $R^C(G) = \bigcap_{k=0}^{\infty} \mu^k(S)$ such that $\mu^k(S) = \times_{i=1}^{I} \mu_i^k(S)$ with

$$\mu_i^k(S) = \bigcup_{\sigma_{-i} \in \triangle\left(\mu_{-i}^{k-1}(S)\right)} f_i(\sigma_{-i})$$

and $\mu_{-i}^0(S) = S_{-i}$.

Proposition: *Suppose all U_i are strictly quasiconcave in S_i or have decreasing utility differences in S_{-i}. Furthermore, assume that the functions $\partial U_i / \partial s_i$ exist, with continuous partial derivatives on S, such that*

$$\partial U_i / \partial s_i \left(f_i(t_{-i}), t_{-i}\right) = 0 \quad \textit{and} \quad \partial^2 U_i / (\partial s_i)^2 \left(f_i(t_{-i}), t_{-i}\right) \neq 0$$

for all $t_{-i} \in S_{-i}$. Then there exists a unique correlated rationalizable strategy for a Cournot oligopoly if one of the following conditions is satisfied

1. For all i and $t_{-i} \in S_{-i}$

$$\sum_{j \neq i} \left| \frac{-\left(\frac{\partial^2 P_i}{\partial s_i \partial s_j}\left(f_i(t_{-i}), t_{-i}\right) * f_i(t_{-i}) + \frac{\partial P_i}{\partial s_j}\left(f_i(t_{-i}), t_{-i}\right)\right)}{\frac{\partial^2 P_i}{(\partial s_i)^2}\left(f_i(t_{-i}), t_{-i}\right) * f_i(t_{-i}) + 2 * \frac{\partial P_i}{\partial s_i}\left(f_i(t_{-i}), t_{-i}\right) - \frac{\partial^2 C_i}{(\partial s_i)^2}\left(f_i(t_{-i})\right)} \right| < 1 \tag{3.1}$$

2. For all j and $t_{-i} \in S_{-i}$

$$\sum_{i \neq j} \left| \frac{-\left(\frac{\partial^2 P_i}{\partial s_i \partial s_j}\left(f_i(t_{-i}), t_{-i}\right) * f_i(t_{-i}) + \frac{\partial P_i}{\partial s_j}\left(f_i(t_{-i}), t_{-i}\right)\right)}{\frac{\partial^2 P_i}{(\partial s_i)^2}\left(f_i(t_{-i}), t_{-i}\right) * f_i(t_{-i}) + 2 * \frac{\partial P_i}{\partial s_i}\left(f_i(t_{-i}), t_{-i}\right) - \frac{\partial^2 C_i}{(\partial s_i)^2}\left(f_i(t_{-i})\right)} \right| < 1 \tag{3.2}$$

Proof: Proposition 4 in chapter 2 of this thesis implies uniqueness of the point-rationalizable solution if $\sum_{j\in I}\left|\frac{\partial f_i}{\partial s_j}(t)\right|<1$ for all $t\in S$ and $i\in I$, or if $\sum_{i\in I}\left|\frac{\partial f_i}{\partial s_j}(t)\right|<1$ for all $t\in S$ and $j\in I$. Let $F(t)=\frac{\partial U_i}{\partial s_i}(t)$, and by application of the Implicit Function Theorem $\left|\frac{\partial f_i}{\partial s_j}(t_{-i})\right|=\left|-\frac{\partial F/\partial s_j}{\partial F/\partial s_i}(f_i(t_{-i}),t_{-i})\right|$. Carrying out the differentiations and setting

$$\sum_{j\in I}\left|\frac{\partial f_i}{\partial s_j}(t)\right|=\sum_{j\neq i}\left|-\frac{\partial F/\partial s_j}{\partial F/\partial s_i}(f_i(t_{-i}),t_{-i})\right|$$

gives condition (3.1). Analogously for condition (3.2). Quasiconcavity, respectively decreasing utility differences, guarantee that a unique point-rationalizable strategy must be the unique correlated rationalizable strategy if all U_i satisfy monotonicity with respect to FOSD, (see proposition 1 and proposition 4 in chapter 5 of this thesis)$\square$

3.3 Model-Specifications

The conditions (3.1) and (3.2) are easily checked for linear inverse demand-functions, i.e., $\frac{\partial^2 P_i}{\partial s_i \partial s_j}(t)=0$ for all $t\in S$. I will speak of a 'negative externality' (NE) Cournot oligopoly if

$$P_i(s)=\max\left\{0,\left(1-\sum_{j\neq i}e_{ij}s_j-s_i\right)\right\}$$

with $e_{ij}\geq 0$, for all $i\neq j$. If $e_{ij}=1$, for all $i\neq j$, the standard Cournot oligopoly with linear demand function and constant marginal cost obtains as a special case of the NE-Cournot oligopoly. But whenever $e_{ij}\neq 1$ the product of the firm j is not anylonger a perfect substitute for the product of firm i, i.e., the goods are not perfectly homogenous; the markets are different.

Such an economy can be interpreted as competition on I different product-markets such that each firm i makes profits on its own product market while it may suffer from a negative externality by the output of a firm j. The externality of firm j's output decision on the profit of firm i is then measured by the 'externality-weight' e_{ij}.

If $e_{ij} = 0$, for all $j \neq i$, the product-market of firm i is not influenced by the output-decisions of the other firms at all, i.e., the firm i has a perfect monopoly for it's own product. Because real-life firms compete rarely on markets of completely homogeneous goods I would expect that the introduction of externality-weights may add a lot of realistic appeal to models of Cournot competition.

Corollary 1: *Suppose $C_i(s_i) = c_i s_i$ with $c_i > 0$. Then there exists a unique correlated rationalizable solution of a NE-Cournot oligopoly if $\sum_{j \neq i} e_{ij} < 2$ for all $i \in I$, or $\sum_{i \in I} e_{ij} < 2$ for all $j \neq i$.*

Proof: The individual best response function f_i for this NE-Cournot oligopoly is given by

$$\max\left\{0, \frac{1 - \sum_{j \neq i} e_{ij} s_j - c_i}{2}\right\}$$

Observe at first that the proposition's assumption of a differentiable best response function is typically not satisfied because there are 'kinks' in f_i at points t_{-i} for which

$$\frac{1 - \sum_{j \neq i} e_{ij} t_j - c_i}{2} = 0$$

However, since the l.h. and the r.h. derivatives exist at these kinks, being either $\sum_{j \neq i} e_{ij}/2$ or zero, the contraction-argument behind the proposition (compare proposition 4 in chapter 2 of this thesis) can be extended without problems to these best response functions. Finally, observe that U_i is strictly concave in S_i and therefore strictly quasiconcave. $\square$

By corollary 1 there exist externality weights greater zero for any number of firms such that the correlated rationalizable strategy of a NE-Cournot oligopoly is unique. Consequently, Bernheim's 'pessimistic observation' depends strongly on his assumption $e_{ij} = 1$ for all e_{ij}. A simple sufficiency condition for uniqueness would be $e_{ij} < 2/\left(I-1\right)$ for all e_{ij}. Hence, for Bernheim's example with three firms we obtain already uniqueness if the assumption of homogenous goods is just slightly relaxed: if the externality weights are smaller than one there exists a unique rationalizable solution for this NE-Cournot oligopoly with three firms.

Corollary 1 shows also that a larger number of firms requires weaker negative externalities if the rationalizable solution shall be unique. Heuristically we can say: the more competition, either from the number of competitors or from the impact of the competitors' products on a firm's 'home market', the more unpredictable is the market outcome by the rationalizability approach.

Corollary 2. *Suppose $C_i\left(s_i\right) = c_i s_i^{\gamma}$ with $\gamma > 1$ and $c_i > 0$. For arbitrary externality weights and for any number of firms there exists some number c such that the correlated rationalizable strategy of the NE-Cournot oligopoly is unique if $c_i \geq c$ for all i.*

Proof: If $\frac{\partial f_i}{\partial s_j}\left(t_{-i}\right) \neq 0$, (respectively l.h. or r.h. derivatives), condition (3.1) becomes

$$\sum_{j\neq i}\left|\frac{-e_{ij}}{2+\gamma\left(\gamma-1\right)c_i * \left(f_i\left(t_{-i}\right)\right)^{\gamma-2}}\right| < 1 \tag{3.3}$$

Even without solving for a particular individual best response function f_i we know that $0 \leq f_i(t_{-i}) \leq 1$ for all $t_{-i} \in S_{-i}$, and the inequality (3.3) holds for a sufficiently great c_i. Moreover, U_i is strictly quasiconcave in S_i.$\square$

Corollary 2 shows that Bernheim's 'pessimistic observation' depends not only on the homogenous-good assumption, $e_{ij} = 1$, but also on the assumed cost-functions. As an example consider a Cournot oligopoly with $e_{ij} = 1$ for all $i \neq j$ and identical quadratic cost-functions $C_i(s_i) = c * s_i^2$ for all i. Substituting in inequality (3.3) we obtain a unique correlated rationalizable strategy if $c > (I - 3)/2$.

3.4 Concluding Remarks

Model-specifications of Cournot oligopolies are identified that imply a unique rationalizable solution. An application of the findings of this chapter to negative externality Cournot oligopoly shows that Bernheim's pessimistic observation, concerning the multitude of correlated rationalizable strategies, depends strongly on the assumed model-parameters.

Two interesting results are obtained. First, for any number of firms the rationalizable solution of a NE-Cournot oligopoly is unique if the negative externality effects between the products of different firms are sufficiently small. Secondly, for any number of firms and for arbitrary externality effects the rationalizable solution of a NE-Cournot oligopoly is unique for appropriate cost-functions.

Börgers and Janssen (1995) investigate Cournot oligopolies for which an increase in the number of firms is appropriately matched by an expansion of the market demand.

Börgers and Janssen show then that the rationalizable solution of sufficiently 'large' Cournot Oligopolies is unique if the according cobweb-process satisfies a particular (rather abstract) stability condition. The specific uniqueness conditions of this chapter can now be applied to determine in turn the stability of the cobweb-process in the 'large' Cournot oligopolies of Börgers and Janssen.

Chapter 4

On The Existence Of Strategic Solutions For Security- And Potential Level Preferences

4.1 Introduction

This chapter investigates the existence of equilibria and of rationalizable strategies in finite games when players have so-called security and potential level preferences over lotteries. Equilibrium concepts, on the one hand, and rationalizability concepts, on the other hand, stand for two different approaches to solve a game, i.e., of predicting how individuals will decide in a situation of strategic interdependency.

The equilibrium approach claims that any solution has to be a Nash equilibrium such that each player of the game chooses a strategy that is a best response against the strategies chosen by his opponents.

The rationalizability approach starts out with the assumption that a player will only choose strategies which are best responses against some strategy choice of his opponents. This assumption may effectively eliminate some strategies as 'unreasonable' and in a next step Rationalizability assumes that a player will only choose best responses against the remaining strategy choices of his opponents. Iteration of this argument gives us finally the set of 'rationalizable' strategies which may be chosen by a player according to the rationalizability approach.

An important question for equilibrium- and for rationalizability concepts concerns the existence of solutions: if there is no Nash equilibrium, respectively no rationalizable strategy, no predictions about players' strategy choices can be made. Moreover, the non-existence of a solution according to a solution concept for a specific game raises the question whether this solution concept is appropriate for solving games in general.

Rationalizability concepts are weaker than equilibrium concepts such that the existence of a Nash equilibrium implies the existence of rationalizable strategies. On the other hand, due to this weakness, rationalizable strategies may exist even if the existence of Nash equilibria fails. For example, in the following version of the 'Matching-Pennies' game, where player A may choose between 'up' and 'down' and player B between 'left' and 'right', every strategy is rationalizable but there does not exist a Nash equilibrium

	left	*right*
up	1,0	0,1
down	0,1	1,0

One way to re-establish here the existence of equilibria is the extension of the strategy sets by 'mixed' strategies: additional to his two 'pure' strategies each player can now also choose among arbitrary randomizations between pure strategies. Another way is the re-interpretation of Nash equilibria as equilibria in beliefs: an equilibrium point is not anylonger defined as mutual best response strategies but as mutual correct beliefs about best response strategies. For the existence of equilibria in beliefs it is not necessary that the players actually randomize between strategies; in the Matching Pennies game we have already an equilibrium in beliefs if both players believe that their opponent chooses one of his pure strategies with equal chance. When we introduce mixed strategies, or when we define equilibria as equilibria in beliefs, we know from Nash (1950a; 1950b) that an equilibrium exists in any game with finite pure strategy sets.

Thus, by Nash's result, existence of equilibria, and therefore existence of rationalizable strategies, appears to be no problem for finite games. However, to work with mixed strategies, or to work with equilibria in beliefs, makes it necessary to work with preferences of players over all probability distributions on the outcomes in a game. Nash's existence proof makes now a very specific assumption about these preferences over probability distributions: all players are assumed to be 'Expected Utility maximizers'. For Expected Utility maximizing players preferences over probability distributions are representable as sums over utility numbers, assigned to the outcomes, and weighed with the probability by which the outcome realizes. This simple mathematical representation of preferences is technically very convenient, and for a long time game-theoretic models have only considered players that are EU-maximizers. But real individuals systematically violate EU-maximizing behavior

(Allais 1953), and several models of preferences over probability distributions have been developed in decision theory with the aim to avoid the descriptive flaws of EU theory (for an overview see, e.g., Starmer 2000; Karni and Schmeidler 1991).

Recently, Crawford (1990) has investigated the existence of equilibria in finite games for players who are not necessarily EU-maximizers. He shows that there may not exist Nash equilibria in mixed strategies but that there exist always equilibria in beliefs as long as the players' preferences are representable by continuous utility functions.

However, some Non-EU models, like the models for 'security and potential level preferences' (Gilboa 1988; Jaffray 1988; Cohen1992; Essid 1997; Schmidt and Zimper 2003), require discontinuous utility representations such that Crawford's existence results for equilibria do not apply to players with SL,PL-preferences. The results of this chapter will show that the existence of equilibria, even defined as equilibria in beliefs, may fail for players with SL,PL-preferences whereas rationalizable strategies always exist. Thus, with SL,PL-preferences it can be impossible that all players choose a best response and have **also** correct anticipations about their opponents' strategy choices.

This situation is similar to the problem of a non-existing equilibrium in the 'Matching Pennies' game if we restrict ourselves to pure strategies and if we define an equilibrium in actively chosen strategies and not in beliefs. But in contrast to 'Matching Pennies' there is now no way to re-establish existence of equilibria by the introduction of mixed strategies or/and by a re-definition as equilibria in beliefs.

On the other hand, there is no problem with the existence of rationalizable strategies, neither in the original 'Matching Pennies' game nor in any finite game of players with SL,PL-preferences. Claiming that the solution of a game must be an equilibrium may make sense for many good reasons in many games. However, when we claim that the equilibrium approach is the only appropriate way of solving a game we can in general not solve games for players with SL,PL-preferences.

The relevance of this chapter's existence, respectively non-existence, results depends on the relevance of the security and potential level preference models; especially, on their assumption on the occurrence of discontinuities in the preferences. Consider a 'Pre-emptive Strike' game where player A can either wag a 'war' or keep 'peace' while a player B may 'destroy' or 'distribute' weapons of mass destruction. Assume the following payoff-matrix for player A

	destroy	*distribute*
peace	2	0
war	1	1

If A expected B to 'destroy' WMD he would choose 'peace', whereas he would choose 'war' if he expected B to 'distribute' WMD. If A is an individual 'who does not take any chances', he would go to 'war' whenever he believes that B 'distributes' WMD with some positive chance. But then A's preferences are not anylonger continuously representable: if the probability of 'distributing' drops to zero there would be an upward-jump in A's evaluation of keeping 'peace'. Security and potential level preference models can take account of such an upward-jump, (in the terminology of the model: a 'security level effect'), and a judgement on the relevance of these models would be (also) a judgement on the relevance of players 'who do not take chances'.

The remainder of this chapter is organized as follows: In section 2 I discuss further the motivation for SL,PL-preferences; especially their aim to provide a good explanation for Allais paradoxa. Furthermore, the discontinuous utility-representation of SL,PL-preferences with threshold values is introduced and it is shown that there do not necessarily exist preference-maximizing lotteries. In section 3 existence of rationalizable strategies for SL,PL-preferences is proved. Section 4 explores in some detail the existence of equilibria for continuously representable preferences. Section 5 demonstrates then the non-existence of equilibria in beliefs for SL,PL-preferences. Section 6 is dedicated to a (at a first glance) curious side result: despite possible non-existence of Nash equilibria there exist always trembling-hand (Selten 1975) and proper equilibria (Myerson 1978) for a particular class (zero-threshold values) of SL,PL-preferences; an interpretation for this existence result is offered. Section 5 concludes. All formal proofs are relegated to the appendix.

4.2 Security- And Potential Level Preferences

4.2.1 SL,PL-Preferences and Allais Paradoxa

Jaffray (1988) presented an axiom system and a representation theorem for preferences over lotteries with the aim to explain Allais paradoxa (see Allais 1953) by the assumption that humans evaluate strongly 'security' ('safety') when confronted with risky decisions; (for a psychological discussion about the relevance of 'security' and 'potential' factors in risky decisions see Lopes 1987). Consider a 'Pre-emptive Strike II' game which extends the above game by the possibility that B has already distributed WMD

	destroy	*distribute*	*already distributed*
peace	2	0	0
war	1	1	0

Suppose player A in the Pre-emptive Strike II game does not take any chances such that he chooses 'war' when he believes B did not yet 'distribute' but might do it with 0.1 chance in the future. How would player A decide when he learnt somehow that player B has 'already distributed' WMD with a high chance, say 0.9? If A's preferences satisfy the so-called independence axiom, valid for Expected Utility theory and for Lexicografic Expected Utility theory (Hausner 1954), he must still prefer 'war' over 'peace'. In contrast, if A goes now for 'peace', and not for 'war', he violates the independence axiom by committing a so-called Allais paradox.

The model of Jaffray allows for this Allais paradox; as does a similar model, independently developed, by Gilboa (1988). Even more importantly, the models of Jaffray and of Gilboa offer a good explanation why A might prefer 'peace' after learning his new information: Because A can not secure himself anylonger from the worst-case scenario 'distribution of WMD' he might as well change his perspective on the problem such that the negative aspects of wagging a 'war' become now more relevant.

Formally, to each lottery a 'security level' is associated (e.g., the worst outcome in the support of a lottery). Then it is assumed that a 'secure' lottery may dominate all 'insecure' lotteries located nearby (in the sense of some mathematically defined neighborhood).

This last property implies 'upward-jumps' in the preferences while passing from insecure to secure lotteries such that utility-representations of SL,PL-models have to be discontinuous.

Cohen (1992) has extended the 'security level' models of Jaffray and Gilboa by 'potential levels'. Now a lottery has, in addition to a security level, some potential level (e.g., the best outcome in the support of the lottery) such that a 'high potential' lottery may dominate all 'low potential' lotteries around. Besides security considerations a decisionmaker may now especially favor the opportunity for excellent outcomes when confronted with risky decisions.

Subsequently, these models have been further extended. Essid (1997) restates Cohen's model for general probability spaces while allowing for infinite outcome-sets. In Schmidt and Zimper (2003) the possibility of positive threshold-probabilities for security- and potential levels is added to Cohen's model to the effect that a small likelihood of a bad or of a good outcome does not necessarily influence the security, respectively potential, level of a lottery.

4.2.2 Utility Representation of SL,PL-Preferences

Throughout this chapter I consider utility representations for a particular class of SL,PL-preferences with positive thresholds discussed in Schmidt and Zimper (2003). Let $X = \{x_1, ..., x_n\}$ denote a finite set of totally ordered deterministic outcomes, and let $\triangle(X)$ denote the set of probability distributions (=lotteries) over X. A lottery $\sigma \in \triangle(X)$ assigns the probability σ_k to an outcome x_k, and δ_x stands for a degenerated lottery that assigns probability one to outcome x. Let $F[\sigma](x_j)$ denote the cumulative distribution function of lottery σ evaluated at outcome x_j.

For so-called 'threshold-values' $\epsilon, \eta \in [0,1)$, with $\epsilon + \eta < 1$, denote by $\Pi(\epsilon,\eta)$ a collection of sets

$$\Pi(\epsilon,\eta) = \{\triangle(j,k)\}_{j=1,\ldots,n;k\geq j}$$

such that

$$\sigma \in \triangle(j,k) \text{ iff } F[\sigma](x_{j-1}) \leq \epsilon,\ F[\sigma](x_j) > \epsilon \text{ AND } 1 - F[\sigma](x_k) \leq \eta,\ 1 - F[\sigma](x_{k-1}) > \eta$$

It can be easily shown (Schmidt and Zimper 2003) that $\Pi(\epsilon,\eta)$ is a partition of $\triangle(X)$ with non-empty cells $\triangle(j,k)$. The threshold-value ϵ implies that worse outcomes than x_j can realize for a lottery of security level j at most with probability ϵ. Accordingly, better outcomes than x_k can realize for a lottery of potential level k at most with probability η. For $\epsilon, \eta = 0$ the partition $\Pi(\epsilon,\eta)$ reduces to the original SL,PL-partition of Cohen (1992), i.e., $\sigma \in \triangle(j,k)$ iff $x_j = \min Support(\sigma)$ and $x_k = \max Support(\sigma)$.

On an axiomatic level SL,PL-models assume basically that the EU-axioms remain valid within identical security -and potential levels subsets whereas a violation of the independence axiom or of the continuity assumption may occur between different subsets; (for detailed axiomatic foundations of SL,PL-models the reader is referred to the literature). Define $\bar{U}_i : \triangle(X) \times \Pi(\epsilon,\eta) \to R^+$ by

$$\bar{U}_i(\sigma, \triangle(j,k)) = m(\triangle(j,k)) + c(\triangle(j,k)) \sum_{k=1}^{n} \sigma(x_k)\, u(x_k)$$

with $m : \Pi(\epsilon,\eta) \to R^+$, $c : \Pi(\epsilon,\eta) \to R^+$, and $u : X \to R^+$.

Definition: *Suppose an individual i has preferences over $\triangle(X)$ that are representable by some utility function $U_i : \triangle(X) \to R^+$ such that $U_i(\sigma) = \bar{U}_i(\sigma, \triangle(j,k))$ for*

$\sigma \in \triangle(j,k)$. Moreover, if

$$m(\triangle(j,k)) + c(\triangle(j,k)) \geq m\left(\triangle\left(j',k'\right)\right) + c\left(\triangle\left(j',k'\right)\right)$$

for $j \geq j'$ and $k \geq k'$, and if $u(x_{m+1}) > u(x_m)$ for $m = 1,...,n-1$, then individual i has SL,PL-preferences over $\triangle(X)$.

A so-called 'security level effect' occurs if and only if there are $\triangle(j',k), \triangle(j,k) \in \Pi(\epsilon,\eta)$, with $j > j'$, such that $m(\triangle(j,k)) + c(\triangle(j,k)) > m(\triangle(j',k)) + c(\triangle(j',k))$. Suppose every open neighborhood around a secure lottery $\sigma \in \triangle(j,k)$ contains insecure lotteries $\sigma' \in \triangle(j',k)$. If there is a SL-effect then there must exist some neighborhood such that σ is strictly preferred over all insecure lotteries in this neighborhood. Accordingly, a 'potential level effect' occurs if there are $\triangle(j,k'), \triangle(j,k) \in \Pi(\epsilon,\eta)$, with $k' > k$, such that $m(\triangle(j,k')) + c(\triangle(j,k)) > m(\triangle(j,k)) + c(\triangle(j',k))$. Here, a low potential lottery $\sigma \in \triangle(j,k)$ would be dominated by all high potential lotteries $\sigma' \in \triangle(j,k')$ that are sufficiently close.

4.2.3 Non-Existence of Preference-Maximizing Lotteries

SL-effects and PL-effects give rise to the discontinuities in the utility representation such that SL,PL-preferences would coincide with EU-preferences without any SL- and PL-effects , i.e.,

$$m(\triangle(j,k)) + c(\triangle(j,k)) = m\left(\triangle\left(j',k'\right)\right) + c\left(\triangle\left(j',k'\right)\right)$$

for all $\triangle(j,k), \triangle(j',k') \in \Pi(\epsilon,\eta)$. Because SL-effects imply 'upper-semicontinuous' utility functions, whereas PL-effects imply 'lower-semicontinuous' utility functions, both kinds of discontinuities have a different impact on the existence of preference-maximizing lotteries on compact subsets of $\triangle$.

Proposition 1. *If there does not occur any potential level effect then there exist preference-maximizing lotteries on each compact subset of* $\triangle$. *But if there occurs a potential level effect and if there exist a high potential lottery* $\sigma' \in \triangle(j,k')$ *and a low potential lottery* $\sigma \in \triangle(j,k)$ *such that the low potential lottery is preferred over the high potential lottery then no preference maximizing lottery exists for some compact subsets of* $\triangle$.

The existence of some high potential lottery that is dominated by some low potential lottery allows for the construction of a sequence of high potential lotteries that converges to a low potential lottery such that each lottery in this sequence is strictly preferred over all preceding lotteries in the sequence. Because the PL-effect implies a lower-semicontinuous utility function the utility drops at the limit point of the sequence and there does not exist a preference maximizing lottery.

The non-existence of preference-maximizing lotteries due to PL-effects can imply the non-existence of best-responses against some belief if a player can arbitrarily randomize between pure strategies. We could therefore expect some impact of proposition 1 on the existence of equilibria as well as on the existence of rationalizable strategies, however, the next section shows that proposition 1 causes no problem for the existence of rationalizable strategies.

4.3 Existence Of Rationalizable Strategies

For a finite set of players I, let S_i denote the finite individual strategy set of player $i \in I$. Let $\triangle^I(S_{-i}) = \times_{j \neq i} \triangle(S_j)$ denote the set of probability distributions over S with independently distributed S_i. An element $\beta_i \in \triangle^I(S_{-i})$ will be called a 'belief' of player i about the strategy choices of his opponents. An element $\sigma_i \in \triangle(S_i)$ denotes a mixed-strategy of player i. Suppose that there exists for each player a preference ordering over the set $\triangle(S_i) \times \triangle^I(S_{-i})$ that can be represented by a utility function $V_i : \triangle(S_i) \times \triangle^I(S_{-i}) \to R$, (if (σ_i, β_i) assigns probability one to some pure strategy-profile s I simply write $V_i(s)$ instead of $V_i(\sigma_i, \beta_i)$).

Let $G = (\triangle(S_i), V_i)_{i \in I}$ denote a finite game in normal form, and recall the definition of an individual best response correspondence in mixed strategies as a mapping $f_i : \triangle^I(S_{-i}) \to 2^{\triangle(S_i)}$ such that

$$f_i(\beta_i) = \left\{ \sigma_i' \mid \sigma_i' \in \arg \max_{\sigma_i \in \triangle(S_i)} V_i(\sigma_i, \beta_i) \right\}$$

Definition (Bernheim 1984; Pearce 1984*): The set of rationalizable mixed-strategies of a game G is given by $R(G) = \bigcap_{k=0}^{\infty} \nu^k(S)$, such that $\nu^k(S) = \times_{i=1}^{I} \nu_i^k(S)$ with*

$$\nu_i^k(S) = \bigcup_{\beta_i \in \triangle^I\left(\nu_{-i}^{k-1}(S)\right)} f_i(\beta_i)$$

and $\nu_{-i}^0(S) = S_{-i}$.

Suppose there exists for all players i of a game G some utility representation U_i of preferences over $\triangle(X)$ such that there is an outcome mapping $o : S \to X$ with

$V_i(s) = U_i(\delta_x)$ for $o(s) = x$. Then G is a game with 'deterministic pure strategy-profiles'. If a game has deterministic strategy profiles we obtain for a player with SL,PL-preferences on $\triangle(X)$, who has belief β_i, the following utility representation for his mixed strategy choices $\sigma_i \in \triangle(S_i)$

$$V_i(\sigma_i, \beta_i) = m(\triangle(j,k)) + c(\triangle(j,k)) \sum_{k=1}^{n} \sigma(x_k)\, u(x_k)$$

with $V_i(\sigma) = U_i(\sigma)$ such that $\sigma = (\sigma_i, \beta_i) \in \triangle(j,k)$ and $\sigma(x_k) = \sum_{\{(s_i,s_{-i})|o(s_i,s_{-i})=x_k\}} \sigma_i(s_i) * \beta_i(s_{-i})$.

Proposition 2. *Given a finite normal form game G with deterministic pure strategy-profiles. There exist always rationalizable mixed-strategies for a player with SL,PL-preferences, i.e., $R(G) \neq \emptyset$.*

The result of proposition 2 is not trivial because by proposition 1 there may not exist best responses in mixed strategies against all possible beliefs for SL,PL-preferences. The existence of rationalizable strategies is due to the fact that SL,PL-preferences do not violate monotonicity with respect to first order stochastic dominance. As a consequence the proof of proposition 2 can simply proceed by showing that there are always pure strategies among the best responses against point-(=probability one) beliefs. (The 'gambling effect' model of Diecidue et al. (2001) is similar to SL,PL-models because it requires a discontinuous utility representation. But because it violates monotonicity with respect to FOSD existence of rationalizable strategies may fail for players with 'gambling effect' preferences.)

The following example shows that a violation of the 'deterministic pure strategy-profiles'-assumption may cause non-existence of rationalizable mixed-strategy profiles.

Example 1: Suppose $X = \{0, 1, 2\}$ and let $\sigma \in \triangle(X)$ denote a lottery with $\sigma(0) = \sigma(2) = 0.5$. Consider a game G with two players, A and B, with the following outcome-matrix of player A

	b
$a1$	σ
$a2$	1

The game G does not satisfy the assumption of 'deterministic pure strategy-profiles' because the pure strategy-profile $(a1, b)$ gives the lottery σ as outcome. Assume now SL,PL-preferences of player A, with $\epsilon, \eta = 0$, such that $u(0) = 0$, $u(1) = 1.5$, $u(2) = 2$, $m(\triangle(j,k)) = 0$ for all $\triangle(j,k)$, and

$$\begin{aligned} c(\triangle(2,2)) &= c(\triangle(1,2)) = c(\triangle(0,2)) = 1.5 \\ c(\triangle(0,0)) &= c(\triangle(0,1)) = c(\triangle(1,1)) = 1 \end{aligned}$$

Observe that $U_A(\sigma) = U_A(\delta_1)$ and $U_A((1-\lambda)\sigma + \lambda\delta_1) < U_A((1-\mu)\sigma + \mu\delta_1)$ for $\lambda < \mu < 1$. With $V_A((1-\lambda)a1 + \lambda a2, \beta_A) = U_A((1-\lambda)\sigma + \lambda\delta_1)$, for $\beta_A(b) = 1$, we obtain

$$V_A(a2, \beta_A) < V_A((1-\lambda)a1 + \lambda a2, \beta_A) < V_A((1-\mu)a1 + \mu a2, \beta_A)$$

for $0 < \lambda < \mu < 1$, i.e., there does not exist a preference maximizing strategy on $\triangle(S_A)$. Consequently, there exist no rationalizable mixed strategies.

4.4 Existence Of Equilibria For Continuous Preferences

For continuously representable preferences over lotteries Crawford (1992) states two results. First, if the players are randomization prone, i.e., players have quasiconcave preferences, there exist always Nash equilibria in mixed strategies. Second, if players are not randomization-prone then Nash equilibria in mixed strategies do not necessarily exist whereas equilibria in beliefs always exist. Let me explain in some detail these existence results for continuously representable preferences.

Let $\sigma \in \triangle^I(S)$ denote a mixed strategy-profile and recall that a Nash equilibrium in mixed strategies σ' of G is defined as a fixed-point of the best-response correspondence $f : \triangle^I(S) \to 2^{\triangle^I(S)}$, i.e., $\sigma' \in f(\sigma')$. For players with preferences on $\triangle^I(S)$ that are quasiconcave and continuously representable the existence of Nash equilibria in mixed strategies is implied for finite normal form games by Kakutani's fixed-point theorem:

Kakutani's Fixed Point Theorem: *Let Y be a compact and convex subset of R^n. If $h : Y \to 2^Y$ is an upper-hemicontinuous correspondence, and if the set $h(y)$ is nonempty and convex for each $y \in Y$ then there exists a fixed point $y^* \in h(y^*)$.*

The set of independent probability distributions over S, i.e., $\times_{j\in I} \triangle(S_j)$, is a compact and convex subset of R^n (in particular it is a simplex of dimension $n = \#X - 1$). By Berge's Maximum Theorem (Berge 1997) the best response correspondence f is upper-hemicontinuous with non-empty values for continuous utility representations $U_i : \triangle^I(S) \to$

R , $i \in I$. For quasiconcave preferences of a player i we have

$$\begin{aligned} V_i(\sigma_i, \sigma_{-i}) &= V_i(\tau_i, \sigma_{-i}) \\ &\Rightarrow V_i(a\sigma_i + (1-a)\tau_i, \sigma_{-i}) \geq V_i(\tau_i, \sigma_{-i}) \end{aligned}$$

for all $\sigma_i, \tau_i \in \triangle(S_i)$, $\sigma_{-i} \in \times_{j \neq i} \triangle(S_j)$ and $a \in (0,1)$. Quasiconcave preferences imply convex values of $f_i(\sigma_{-i})$: if $\sigma_i, \tau_i \in \triangle(S_i)$ are best responses then all convex-combinations $a\sigma_i + (1-a)\tau_i$, $a \in (0,1)$, have to be best responses as well. Expected Utility preferences are quasiconvex because the Independence-axiom of EU-theory claims that

$$\begin{aligned} V_i(\sigma_i, \sigma_{-i}) &= V_i(\tau_i, \sigma_{-i}) \\ &\Rightarrow V_i(a\sigma_i + (1-a)\rho_i, \sigma_{-i}) = V_i(a\tau_i + (1-a)\rho_i, \sigma_{-i}) \end{aligned}$$

for arbitrary $\rho_i \in \triangle(S_i)$. Thus, Nash's existence result for equilibrium points in mixed strategies for EU-preferences (Nash 1950a) is implied by the existence result for quasiconcave preferences that are continuously representable.

For a game with I players an equilibrium in beliefs is defined as an I-tupel of beliefs such that i.) for each player only best responses of his opponents against their beliefs are in the support of his belief, and ii.) any two players share the same belief concerning the strategy-choice of some third player, (see also Crawford 1990 who defines an equilibrium in beliefs only for two players). Formally, define a 'beliefs over best-responses' correspondence $g : \triangle^I(S) \to 2^{\triangle^I(S)}$ by $g(\beta) = \times_{i \in I} g_i(\beta_i)$ with

$$g_i(\beta_i) = \triangle \left\{ \sigma_i' \mid \sigma_i' \in \arg \max_{\sigma_i \in \triangle(S_i)} V_i(\sigma_i, \beta_i) \right\}$$

Definition: *An equilibrium in beliefs* β' *of a finite normal form game* G *is a fixed-point of the 'beliefs over best-responses' correspondence, i.e.,* $\beta' \in g(\beta')$.

In contrast to $f_i(\sigma_{-i})$, which collects mixed-strategies that are best-responses of player i against the mixed strategy profile σ_{-i}, an element of $g_i(\beta_i)$, the 'beliefs over best-responses' correspondence, should be interpreted as the identical belief of all players $j \neq i$ about player i's best-response choice given his belief β_i. Hence, an equilibrium in beliefs does not refer to mutually satisfied mixed-strategy choices, as the Nash equilibrium, but to mutually satisfied beliefs about mixed-strategy choices.

Mathematically, the value of the individual 'beliefs over best-responses' $g_i(\beta_i)$ is nothing else than the convexification of the value of the individual best-response correspondence $f_i(\sigma_{-i})$ with $\sigma_{-i} = \beta_i$. As one consequence we obtain immediately the equivalence of both equilibrium definitions, i.e., $f_i(\sigma_{-i}) = g_i(\beta_i)$ for $\sigma_{-i} = \beta_i$, for players with quasiconcave preferences: the values of the best response correspondence are already convex.

As another consequence equilibria in beliefs exist by Kakutani's fixed-point theorem as long as the players preferences are continuously representable: Because the 'beliefs-over-best-responses'-correspondence assumes already convex values by definition we can drop the assumption of quasiconcave preferences while applying nevertheless Kakutani's fixed-point theorem.

4.5 Non-Existence Of Equilibria For SL,PL-Preferences

The discussion in section 4 shows that for continuous utility representations the question of existence of equilibria is basically a question of convex-valued correspondences. This is not anylonger true for the discontinuous utility representations required by SL,PL-preferences.

Proposition 3. *For SL,PL-preferences the existence of an equilibrium in beliefs, and therefore of a Nash equilibrium in mixed strategies, may fail in finite normal form games with deterministic pure strategy-profiles.*

Example 2: Non-existence due to PL-effect with positive threshold-value $\eta > 0$. Consider a game with two players, A and B, and the following outcome-matrix (not utility-matrix!) for pure strategy profiles:

	$b1$	$b2$
$a1$	$0,3$	$2,0$
$a2$	$1,0$	$0,1$

Assume that the preferences of player A are representable by a SL,PL-model such that $u(0) = 0$, $u(1) = 1$, $u(2) = 2$, and $c(\triangle(j,k)) = 1$ for all $\triangle(j,k)$, and

$$m(\triangle(2,2)) = m(\triangle(1,2)) = m(\triangle(0,2)) = 1$$

$$m(\triangle(0,0)) = m(\triangle(0,1)) = m(\triangle(1,1)) = 0$$

Thus, there occur PL-effects but no SL-effects. Moreover, assume a threshold-value $\eta = 0.1$ for the PL-effect. The values of the best response correspondence of player

A, i.e., the values of $f_A(\lambda b1 + (1-\lambda) b2)$, are then given as

$$\begin{array}{ll} \{a1\} & 0 \leq \lambda < \frac{2}{3} \\ \{\mu a1 + (1-\mu) a2 \mid 0.3 < \mu \leq 1\} & \lambda = \frac{2}{3} \\ \emptyset & \frac{2}{3} < \lambda < 1-\eta \\ \{a2\} & 1-\eta \leq \lambda \leq 1 \end{array}$$

To see why there do not exist best responses in mixed strategies against beliefs with $\frac{2}{3} < \lambda < 1-\eta$ assume, for example, that $\lambda = 0.7$. The utility

$$V_A(\mu a1 + (1-\mu) a2, \lambda b1 + (1-\lambda) b2)$$

is then strictly increasing in $(1-\mu)$ as long as $\mu > \frac{1}{3}$: if $\mu * (1-\lambda) > \eta$ the mixed strategy profiles are all evaluated at the same potential level. However, for $\mu = \frac{1}{3}$ the probability of the high outcome $x = 2$ reaches the threshold-value $\eta = 0.1$ such that the utility of the strategies with $\mu \leq \frac{1}{3}$ drops sharply due to the decrease in the potential level, (and it does not 'recover' at $\mu = 0$.)

Let player B be an EU-maximizer with $u(x) = x$, and observe that there does not exist an equilibrium in pure strategies: In order to become indifferent between his strategy choices B has to believe that A chooses $a1$ with 0.25 and $a2$ with 0.75 probability, i.e., $\beta_B(a1) = \mu = 0.25$. But as just seen such a belief of player B is not consistent with any best-responses of player A. Consequently, there does not exist an equilibrium in beliefs.

Example 3: Non-existence due to SL-effect with $\epsilon = 0$. Consider the following outcome-matrix

	$b1$	$b2$
$a1$	$1,1$	$2,0$
$a2$	$2,0$	$0,1$

Assume that the preferences of player A are representable by a SL,PL-model with $\epsilon, \eta = 0$ such that $u(x) = x$, $x \in \{0,1,2\}$, $m(\triangle(j,k)) = 0$ for all $\triangle(j,k)$, and

$$\begin{aligned} c(\triangle(1,1)) &= c(\triangle(1,2)) = c(\triangle(2,2)) = 2 \\ c(\triangle(0,0)) &= c(\triangle(0,1)) = c(\triangle(0,2)) = 1 \end{aligned}$$

Now there occur only SL-effects and the values of $f_A(\lambda b1 + (1-\lambda) b2)$ are given as

$$\begin{array}{ll} \{a1\} & 0 \leq \lambda < 1 \\ \{a2\} & \lambda = 1 \end{array}$$

Let player B be an EU-maximizer, and observe that there does not exist an equilibrium in beliefs.

Example 4: Non-existence due to SL-effect with $\epsilon = 0.1$. Consider the following outcome-matrix

	$b1$	$b2$
$a1$	$0,1$	$1,0$
$a2$	$10,0$	$0,1$

Assume that the preferences of player A are representable by a SL,PL-model with $\epsilon = 0.1$ such that $u(x) = x$, $m(\triangle(j,k)) = 0$ for all $\triangle(j,k)$, and

$$c(\triangle(1,1)) = c(\triangle(1,10)) = c(\triangle(10,10)) = 2$$

$$c(\triangle(0,0)) = c(\triangle(0,1)) = c(\triangle(0,10)) = 1$$

The values of the best response correspondence $f_A(\lambda b1 + (1-\lambda) b2)$ are given as

$$\begin{array}{ll} \{a1\} & 0 \leq \lambda \leq 0.1 \\ \{a2\} & 0.1 < \lambda \leq 1 \end{array}$$

As in example 3 there does not exist an equilibrium in beliefs if player B is an EU-maximizer.

Proposition 3 shows that the convex values of the 'beliefs over best-responses' correspondence can not assure the existence of an equilibrium in beliefs for SL,PL-preferences: In the examples 2-4 the values of the best-response correspondences have been already convex. In example 2 non-existence of best responses and in example 3 and 4 non-upperhemicontinuity of the best-response correspondence imply the non-existence of equilibrium in beliefs.

4.6 Existence Of Trembling Hand Equilibria For Zero-Thresholds

By proposition 3 there may not exist equilibria in beliefs, or equilibria in mixed strategies, for players with SL,PL-preferences. The examples show that this non-existence result holds regardless whether the thresholds are positive or zero. However, for SL,PL-preferences with zero thresholds, i.e., $\epsilon, \eta = 0$, the existence of trembling-hand equilibria (Selten 1975) and of proper equilibria (Myerson 1978) can be shown.

Define by $\triangle^{\varepsilon}(S_i) \subset \triangle(S_i)$ a set of mixed strategies such that each $s_i \in S_i$ appears in the support of $\sigma_i \in \triangle^{\varepsilon}(S_i)$ with probability $\sigma_i(s_i) \geq \varepsilon(s_i) > 0$, and let us call $G(\varepsilon) =$

$(\triangle^{\varepsilon}(S_i), V_i)_{i \in I}$ a perturbed game of G. Moreover, for a given game $G = (\triangle(S_i), V_i)_{i \in I}$ define the game $G^{mix} = \left(\triangle(S_i), V_i^{mix}\right)_{i \in I}$ by letting $V_i^{mix}(\sigma) = \bar{U}_i(\sigma, \triangle(1,n))$ for all $\sigma \in \triangle^I(S)$.

Definition (Selten 1975; theorem 2.2.5 in van Damme 1991): *The mixed-strategy profile* $\sigma' = \times_{i \in I} \sigma_i'$ *is called a trembling-hand equilibrium of* G *if and only if there is a sequence* $\{\varepsilon^m\}_{m \in N}$, *with* $\lim_{m \to \infty} \varepsilon^m(s_i) = 0$ *for all* $s_i \in S_i$ *and all* $i \in I$, *such that* $\sigma' = \lim_{m \to \infty} \sigma'(\varepsilon^m)$ *with* $\sigma'(\varepsilon^m)$ *being a Nash equilibrium of the perturbed game* $G(\varepsilon^m)$.

Proposition 4. *A trembling-hand equilibrium* σ' *exists for every finite normal form game* G *with deterministic pure strategy-profiles if the players have SL,PL-model preferences with* $\epsilon, \eta = 0$. *Moreover,* σ' *is a trembling-hand equilibrium of* G *if and only if* σ' *is a trembling-hand equilibrium of* G^{mix}.

Proposition 4 is easily derived: For every perturbed game $G(\varepsilon)$ players with SL,PL-preferences and zero-thresholds behave like EU-maximizers: because every outcome realizes with positive probability all relevant mixed-strategy profiles are evaluated within the same subset $\triangle(1,n)$. As a consequence there exists a Nash equilibrium for every perturbed game $G(\varepsilon^m)$, and the limit point of the Nash equilibria for some sequence of perturbed games is exactly a trembling-hand equilibrium of G^{mix}.

Obviously, the same argumentation goes through for the proper equilibrium of Myerson (1978) who imposes particular restrictions on the probability weights $\varepsilon(s_i)$ by which pure strategies s_i may be played in a so-called ε-proper equilibrium $\sigma'(\varepsilon)$ of G. In particular, it is required for an ε-proper equilibrium $\sigma'(\varepsilon)$ of G that $0 < \varepsilon(t_i) = \sigma'_i(t_i) \leq \varepsilon(s_i) * \varepsilon(s_i)$ if $V_i\left(t_i, \sigma'_{-i}(\varepsilon)\right) < V_i\left(s_i, \sigma'_{-i}(\varepsilon)\right)$. A proper equilibrium of G is then defined as a limit point $\sigma' = \lim_{m\to\infty} \sigma'(\varepsilon^m)$ with each $\sigma'(\varepsilon^m)$ being an ε-proper equilibrium of G and $\lim_{m\to\infty} \varepsilon_m(s_i) = 0$ for all $s_i \in S_i$ and all $i \in I$. Analogously, a proper equilibrium of G is given by a proper equilibrium of G^{mix}; which exists actually (Myerson 1978; van Damme 1991).

The concepts of trembling-hand equilibria or of proper equilibria do not work here anylonger as a selection ('perfectness') criterion for Nash equilibria in finite normal form games. For example, the unique trembling-hand equilibrium σ' in the game of example 3 is given by $\sigma'_A(a1) = \frac{1}{2}$ and $\sigma'_B(b1) = \frac{2}{3}$. But this mixed-strategy profile is not a Nash equilibrium. Therefore a trembling-hand equilibrium can only be interpreted as an 'approximation'-result: We have to assume that the players are actually playing a slightly perturbed game such that a trembling-hand equilibrium approximates for an observer the Nash equilibrium played in the perturbed game which is not exactly known by this observer. The assumption of a slightly perturbed game which is known to the players but not to the modeler may have some appeal in particular situations but it is also questionable as a general doctrine.

4.7 Concluding Remarks

A Nash equilibrium in mixed strategies exists in finite normal form games for players who are randomization-prone and whose preferences are continuously representable. For players who are not randomization-prone, i.e., who have not quasiconcave preferences, the existence of a Nash equilibrium in mixed strategies may fail due to non-convex values of the best-response correspondence. However, if the preferences of these players are continuously representable then there exists an equilibrium in beliefs for each finite normal norm game.

For the discontinuous utility representations of SL,PL-preferences we obtain for finite normal form games that there does not necessarily exist a best response in mixed strategies against each belief, and that the best-response correspondence is not necessarily upper-hemicontinuous. As a consequence it is not difficult to find examples of finite normal form games with SL,PL-preferences such that there does not exist an equilibrium in beliefs.

Somewhat surprisingly existence of trembling-hand equilibria (Selten 1975) and of proper equilibria (Myerson 1978) can be established for the particular class of SL,PL-preferences with zero-thresholds. Because trembling-hand, or proper, equilibria are not necessarily Nash equilibria for these games the value of this existence result is in my opinion limited. If we assume that the players are playing indeed some perturbed game, while we are only aware of the unperturbed version, we can take the trembling-hand equilibrium as an approximation for the Nash equilibrium in the perturbed game. Besides this approximation of Nash equilibria in perturbed games I do not see any other meaningful interpretation of trembling-hand, or proper, equilibria that are not Nash equilibria.

In contrast to equilibrium concepts rationalizability does rather well as solution concept: There exists always rationalizable mixed strategies in finite normal form games for players with SL,PL-preferences. (Though there may not exist best-responses in mixed strategies against all possible beliefs.) From an applicational point of view the rationalizability approach is often considered as a less attractive than the equilibrium approach because it is a weaker solution concept. However, the findings of this chapter cast some doubt on the usefulness of the equilibrium approach if we consider players with SL,PL-preferences: While there exist always rationalizable strategies it may be impossible that all players choose best responses and have also correct anticipations about their opponents' strategy choices.

4.8 Appendix: Proofs

Proof of proposition 1: There exist preference-maximizing lotteries for each compact subset of $\triangle$ if U_i is upper-semicontinuous on $\triangle$, i.e., for any converging sequence with $\lim_{n\rightarrow\infty}\sigma_n=\sigma$

$$\limsup U_i(\sigma_n)\leq U_i(\sigma) \tag{4.1}$$

Observe that there exist only sequences $\{\sigma_n\}_{n\in N}$ that converge to some lottery σ with the same, or with a higher, security level than the members of the sequence σ_n with $n\geq M$ for some finite M. Consequently, if there does not occur a PL-effect then condition (4.1) must be satisfied; (if σ has a higher security level than σ_n with $n\geq M$ then (4.1) holds with strict inequality, and with equality else). Consider now a PL-effect such that $U_i(\sigma')<\bar{U}_i(\sigma',\triangle(j,k'))$ for some $\sigma'\in\triangle(j,k)$ with $k'>k$, and assume there exist lotteries $\sigma\in\triangle(j,k)$ and $\tau\in\triangle(j,k')$ with $U_i(\tau)\leq U_i(\sigma)$. Observe that there exists for each $\Pi(\epsilon,\eta)$

a unique $\lambda^* \in (0,1]$ such that $(1-\lambda^*)\,\tau + \lambda^*\sigma \in \triangle(j,k)$ and $(1-\lambda)\,\tau + \lambda\sigma \in \triangle(j,k')$ for $0 \le \lambda < \lambda^*$, (e.g., if $\eta = 0$ then $\lambda^* = 1$). By definition of a PL-effect we have

$$\bar{U}_i\left((1-\lambda^*)\,\tau + \lambda^*\sigma, \triangle\left(j,k'\right)\right) > U_i\left((1-\lambda^*)\,\tau + \lambda^*\sigma\right)$$

Since $U_i(\sigma) < \bar{U}_i(\sigma, \triangle(j,k'))$ the relation $U_i(\tau) \le U_i(\sigma)$ implies

$$\bar{U}_i\left(\tau, \triangle\left(j,k'\right)\right) < \bar{U}_i\left(\sigma, \triangle\left(j,k'\right)\right)$$

and we obtain

$$U_i\left((1-\lambda)\,\tau + \lambda\sigma\right) < U_i\left((1-\mu)\,\tau + \mu\sigma\right)$$

for $0 \le \lambda < \mu < \lambda^*$. As a consequence there does not exist any preference-maximizing lottery on the compact subset

$$\left\{(1-\lambda)\,\tau + \lambda\sigma \mid \lambda \in [0,\lambda^*]\right\} \subset \triangle$$

□

Proof of proposition 2: Take some point-belief $\beta_i(s_{-i}) = 1$ and pick a pure strategy s_i' that is preference-maximizing over S_i, i.e., $s_i' \in \arg\max_{s_i \in S_i} V_i(s_i, \beta_i)$, (since S_i is finite the existence of s_i' is guaranteed). Suppose now there exists some mixed strategy $\sigma_i' \in \triangle(S_i)$ such that $V_i(\sigma_i', \beta_i) > V_i(s_i, \beta_i)$. But this is impossible: Since x_k, given by $o(s_i', s_{-i}) = x_k$, is the maximal element in the support of any lottery (σ_i, β_i) we have $(\sigma_i, \beta_i) \in \triangle(j', k')$ with $j' \le k$ and $k' \le k$ for all $\sigma_i \in \triangle(S_i)$, which implies for SL,PL-preferences

$$\begin{aligned} V_i\left(s_i', \beta_i\right) &= U_i(\delta_k) = \bar{U}_i(\delta_k, \triangle(k,k)) \\ &\ge \bar{U}_i(\sigma_i, \beta_i, \triangle(k,k)) \ge U_i\left(\sigma_i, \beta_i, \triangle\left(j', k'\right)\right) = V_i(\sigma_i, \beta_i) \end{aligned}$$

Hence, s_i' must be a best response against the point-belief β_i on $\triangle(S_i)$. As a consequence the set of point-rationalizable solutions in mixed-strategies (compare Bernheim 1984) is non-empty; which implies non-emptiness of $R(G)$.$\square$

Chapter 5

Equivalence Conditions For Rationalizability Concepts

5.1 Introduction

Rationalizability, point-rationalizability, and correlated rationalizability (Bernheim 1984; Pearce 1984) can be interpreted as solution concepts for normal form games that combine the standard approach of decision theory with a particular epistemic assumption (Tan and Werlang 1988). First, it is assumed that each player resolves his uncertainty about his opponents' strategy choices by a unique 'belief', i.e., a probability distribution over the opponents' strategy choices. Secondly, the class of possible beliefs is restricted by the epistemic assumption that it is "common knowledge among the players that only best responses against some belief are chosen".

While they share the same epistemic assumption these three rationalizability concepts differ in their restrictions on the initial class of admissible beliefs by which a player may resolve his uncertainty: point-rationalizability considers only "probability one" beliefs (=point-beliefs), correlated rationalizability allows for arbitrary beliefs, and the standard definition of rationalizability assumes beliefs over independent strategy choices. The restriction to degenerated probability distributions by point-rationalizability is not very convincing and it may imply unreasonable results. Consider, for example, the following matrix of utility numbers for EU-maximizing players A and B

	$b1$	$b2$
$a1$	1 , 0	-100 , 0
$a2$	0.99 , 0	0.99 , 0
$a3$	-100 , 0	1 , 0

The strategy $a2$ is not point-rationalizable because it is not a best response of A against any pure strategy of B. Nevertheless, $a2$ appears as a reasonable choice of A given that B is indifferent between his strategy choices.

The big advantage of point-rationalizability is its technical convenience: point-rationalizability considers only best responses against pure strategies whereas the other rationalizability concepts must consider best responses against (non-degenerated) probability distributions over these pure strategies. Thus, whenever the set of strategies differs from the set of probability distributions over strategies it is much easier to work with point-rationalizability than with other rationalizability concepts.

To see the difference between rationalizability and correlated rationalizability consider the following utility-matrix of an EU-maximizing player A.

	$(b1, c1)$	$(b1, c2)$	$(b2, c1)$	$(b2, c2)$
$a1$	1	1	1	0
$a2$	0.5	-100	-100	0.5
$a3$	0	1	1	1

The strategy $a2$ is a best response only when player A believes that player B and player C choose with 0.5 chance either $(b1, c1)$ or $(b2, c2)$. This belief does not assume independently chosen strategies and the strategy $a2$ is therefore not rationalizable. However, it may be correlated-rationalizable.

Whether the standard definition of rationalizability is more appropriate than correlated rationalizability depends from case to case: for some strategic situations the restriction to beliefs over independent strategy choices is a natural assumption, for other strategic situations it is not (compare also Brandenburger and Dekel 1987; Epstein 1997).

This chapter explores conditions for which all three rationalizability concepts are equivalent, i.e., for which the sets of rationalizable, of correlated rationalizable, and of point-rationalizable strategies coincide. If a game satisfies these conditions the technical convenience of point-rationalizability would go along with the interpretational advantage of the other rationalizability concepts. Moreover, for games satisfying these equivalence conditions, the question becomes irrelevant whether the assumption of arbitrary beliefs or of beliefs restricted to independent strategy choices is more appropriate for a given situation.

For instance, in chapter 2 of this thesis I characterize games with a unique point-rationalizable solution by mathematical conditions that are not at hand for the other rationalizability concepts (contraction-properties of a best-response function against pure strategies). In a second step I apply then the equivalence results of this chapter in order to generalize the uniqueness results from point-rationalizability to rationalizability and to correlated rationalizability.

The main findings of this chapter are most useful for two classes of games: i.) the individual strategy sets are compact intervals of the real line, ii.) the individual strategy sets are complete lattices and there exists a unique point-rationalizable strategy.

Ad i.) Quasiconcave utility functions or utility functions with monotonic differences imply equivalence of all three rationalizability concepts.

Ad ii.) If the utility functions are monotonic and 'supermodular' (Topkis 1979; Vives 1990; Milgrom and Roberts 1990) then a unique point-rationalizable strategy is also the unique rationalizable and the unique correlated rationalizable strategy.

On the one hand the second result restates a finding of Milgrom and Roberts (1990) for supermodular games (=increasing utility differences for all players). On the other hand it provides an extension to games for which a player has either decreasing or increasing utility differences. For example, the equivalence results of this chapter apply now also to models of Cournot oligopolies that are not supermodular in the sense of Milgrom and Roberts.

This chapter investigates also equivalence conditions for rationalizability concepts and for different concepts of iterated elimination of dominated strategies. Useful uniqueness results for rationalizable strategies have been indirectly derived via the iterated elimination of dominated strategies (Milgrom and Roberts 1990; Moulin 1984). This is not surprising because these solution concepts offer the same technical advantage as point-rationalizability:

instead of probability distributions over opponents' strategy choices only pure strategies have to be considered. There exist famous results in the literature that relate both families of solution concepts (Pearce 1984; Moulin 1984; Milgrom and Roberts 1990) and I will provide restatements as well as some extensions of these results.

Interestingly, all equivalence results (except the observation in section 3) in this chapter do not require EU-maximizing players: simple stochastic dominance conditions, implied by monotonicity with respect to first order stochastic dominance, are sufficient.

The remainder of this chapter proceeds as follows. In section 2 the notation is introduced and formal definitions of rationalizability concepts and of concepts of iterated elimination of dominated strategies are provided. Section 3 reviews existing equivalence results. Section 4 presents equivalence results for quasiconcave utility functions. In section 5 equivalence results for utility functions with increasing or decreasing differences are derived. In section 6 problems for further generalizations of the obtained equivalence conditions are discussed. Section 7 concludes. All proofs are relegated to the appendix.

5.2 Preliminaries: Notation, Definitions

For a given set of players I, S_i denotes the individual strategy set of player $i \in I$. Let $\triangle(S_{-i})$ denote the set of probability distributions over $S_{-i} = \times_{j \neq i} S_j$. An element $\sigma_{-i} \in \triangle(S_{-i})$ will be called a 'belief' of player i about the strategy choices of his opponents. $\triangle^I(S_{-i})$, with $\triangle^I(S_{-i}) = \times_{j \neq i} \triangle(S_j)$, denotes the set of player i's beliefs restricted to independent strategy choices of his opponents. The preferences of a player i over the set

$S_i \times \triangle(S_{-i})$ shall be representable by some utility function $U_i : S_i \times \triangle(S_{-i}) \rightarrow R$.

Let $G = (S_i, U_i)_{i \in I}$ denote a game in normal form. An individual best response correspondence is a mapping $f_i : \triangle(S_{-i}) \rightarrow 2^{S_i}$ such that $f_i(\sigma_{-i}) = \arg\max_{s_i \in S_i} U_i(s_i, \sigma_{-i})$. In case σ_{-i} is a 'point-belief', i.e., σ_{-i} assigns probability one to some strategy profile s_{-i}, I simply write $f_i(s_{-i})$ instead of $f_i(\sigma_{-i})$. For single-valued $f_i(\sigma_{-i})$ I write $s_i = f_i(\sigma_{-i})$ instead of $s_i \in f_i(\sigma_{-i})$.

If $U_i(s_i, s_{-i}) > U(t_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ implies $U_i(s_i, \sigma_{-i}) > U(t_i, \sigma_{-i})$ for all $\sigma_{-i} \in \triangle(S_{-i})$ player i's preference-ordering satisfies 'monotonicity with respect to strong stochastic dominance' (SSD). When we consider players who may randomize between pure strategies, i.e., an element $\sigma_i \in \triangle(S_i)$ is then a mixed strategy of player i, we need a preference-ordering defined over the set $\triangle(S_i) \times \triangle(S_{-i}) \subset \triangle(S)$. Suppose $U_i(s_i, s_{-i}) \geq U(t_i, s_{-i})$ for all t_i in the support of σ_i and $U_i(s_i, s_{-i}) > U(t_i, s_{-i})$ for some t_i in the support of σ_i. Then player i is 'weakly gambling averse' if his preferences satisfy $U_i(s_i, s_{-i}) > U(\sigma_i, s_{-i})$.

Obviously, monotonicity with respect to SSD and weak gambling aversion are implied by monotonicity with respect to first order stochastic dominance whenever the strategies in S result in deterministic outcomes. As a consequence all monotonicity conditions of this chapter should be satisfied for 'reasonable' players.

Definition *(*Bernheim 1984; Pearce 1984*): The set of rationalizable strategies for a game G is given by $R(G) = \bigcap_{k=0}^{\infty} \nu^k(S)$, such that $\nu^k(S) = \times_{i=1}^{I} \nu_i^k(S)$ with*

$$\nu_i^k(S) = \bigcup_{\sigma_{-i} \in \triangle^I \left(\nu_{-i}^{k-1}(S)\right)} f_i(\sigma_{-i})$$

and $\nu^0_{-i}(S) = S_{-i}$.

Definition *(*Pearce 1984*): The set of correlated rationalizable strategies for a game G is given by* $R^C(G) = \bigcap_{k=0}^{\infty} \mu^k(S)$, *such that* $\mu^k(S) = \times_{i=1}^{I} \mu_i^k(S)$ *with*

$$\mu_i^k(S) = \bigcup_{\sigma_{-i} \in \triangle\left(\mu_{-i}^{k-1}(S)\right)} f_i(\sigma_{-i})$$

and $\mu^0_{-i}(S) = S_{-i}$.

Definition *(*Bernheim 1984; Pearce 1984*): The set of point-rationalizable strategies for a game G is given by* $P(G) = \bigcap_{k=0}^{\infty} \lambda^k(S)$, *such that* $\lambda^k(S) = \times_{i=1}^{I} \lambda_i^k(S)$ *with*

$$\lambda_i^k(S) = \bigcup_{s_{-i} \in \lambda_{-i}^{k-1}(S)} f_i(s_{-i})$$

and $\lambda^0_{-i}(S) = S_{-i}$.

The following inclusions are immediately implied by the above definitions: $P(G) \subset R(G) \subset R^C(G)$. Moreover, notice that $\triangle^I(S_{-i}) = \triangle(S_{-i})$ for $I \leq 2$ such that rationalizability and correlated rationalizability can only differ for games with more than two players.

Let us now turn to different concepts of iterated elimination of dominated strategies which make different assumptions with respect to the definition of a 'dominated' strategy.

Definition:*The weak dominance solution of a game G against pure strategies (*Moulin *1984) is given by* $D^{WP}(G) = \bigcap_{k=0}^{\infty} \delta^k(S)$, *such that* $\delta^k(S) = \times_{i=1}^{I} \delta_i^k$ *whereas* $s_i \in \delta_i^k$ *if and only if* $s_i \in \delta_i^{k-1}$, *and there does not exist a* $t_i \in \delta_i^{k-1}$ *such that* $U(t_i, s_{-i}) \geq$

$U(s_i, s_{-i})$ for all $s_{-i} \in \delta_{-i}^{k-1}$, and $U(t_i, s_{-i}) > U(s_i, s_{-i})$ for some $s_{-i} \in \delta_{-i}^{k-1}$. Furthermore, $\delta_{-i}^{0}(S) = S_{-i}$.

Definition:*The strong dominance solution of a game G against pure strategies is given by $D^{SP}(G) = \bigcap_{k=0}^{\infty} \theta^k(S)$, such that $\theta^k(S) = \times_{i=1}^{I} \theta_i^k$ whereas $s_i \in \theta_i^k$ if and only if $s_i \in \theta_i^{k-1}$, and there does not exist a $t_i \in \theta_i^{k-1}$ such that $U(t_i, s_{-i}) > U(s_i, s_{-i})$ for all $s_{-i} \in \theta_{-i}^{k-1}$. Furthermore, $\theta_{-i}^{0}(S) = S_{-i}$.*

Definition:*The weak dominance solution of a game G against mixed strategies is given by $D^{WM}(G) = \bigcap_{k=0}^{\infty} \psi^k(S)$, such that $\psi^k(S) = \times_{i=1}^{I} \psi_i^k$ whereas $s_i \in \psi_i^k$ if and only if $s_i \in \psi_i^{k-1}$, and there does not exist a $\sigma_i \in \triangle\left(\psi_i^{k-1}\right)$ such that $U(\sigma_i, s_{-i}) \geq U(s_i, s_{-i})$ for all $s_{-i} \in \psi_{-i}^{k-1}$, and $U(\sigma_i, s_{-i}) > U(s_i, s_{-i})$ for some $s_{-i} \in \psi_{-i}^{k-1}$. Furthermore, $\psi_{-i}^{0}(S) = S_{-i}$.*

Definition:*The strong dominance solution of a game G against mixed strategies is given by $D^{SM}(G) = \bigcap_{k=0}^{\infty} \varphi^k(S)$, such that $\varphi^k(S) = \times_{i=1}^{I} \varphi_i^k$ whereas $s_i \in \varphi_i^k$ if and only if $s_i \in \varphi_i^{k-1}$, and there does not exist a $\sigma_i \in \triangle\left(\varphi_i^{k-1}\right)$ such that $U(\sigma_i, s_{-i}) > U(s_i, s_{-i})$ for all $s_{-i} \in \varphi_{-i}^{k-1}$. Furthermore, $\varphi_{-i}^{0}(S) = S_{-i}$.*

5.3 Existing Equivalence Results

In the literature the notion of a 'dominance solution' refers usually to the 'strong dominance solution against mixed strategies', (see Fudenberg and Tirole 1996; Milgrom and Roberts 1990). The above definitions imply $D^{WM}(G) \subset D^{SM}(G) \subset D^{SP}(G)$ and $D^{WM}(G) \subset D^{WP}(G) \subset D^{SP}(G)$. For the relation of rationalizability- and of dominance solution concepts we obtain immediately $P(G) \subset D^{SP}(G)$, and in case monotonicity with respect to SSD is satisfied: $R^{C}(G) \subset D^{SP}(G)$.

Besides these obvious relations there exist, to my knowledge, three important results that provide conditions under which particular rationalizability concepts are equivalent with particular dominance solution concepts. Another equivalence result concerns specific dominance- and rationality concepts proposed by Börgers (1993) which will not play a further role for the results of this chapter. Let me review these existing equivalence results.

By a corollary in Milgrom and Roberts (1990) the unique Nash equilibrium of a supermodular game is also the unique strategy profile in the strong dominance solution against pure strategies. Thus, if monotonicity with respect to SSD is satisfied we have for a supermodular game

$$P(G) = R(G) = R^{C}(G) = D^{SP}(G) = D^{WP}(G)$$

whenever the Nash equilibrium of G is unique. If weak gambling aversion is also satisfied we have moreover $P(G) = D^{WM}(G) = D^{SM}(G)$. This finding will be restated and extended by corollary 1 of this chapter.

The Lemma 2 in Moulin (1984) identifies conditions for a game G such that the 'weak dominance solution against pure strategies' coincides with the set of point-rationalizable strategy profiles, i.e., $D^{WP}(G) = P(G)$. Moulin's result will be restated and extended by proposition 2 of this chapter.

It is well known (at least for finite strategy sets) that the strong dominance solution against mixed strategies is equivalent with the set of correlated rationalizable strategies for Expected Utility maximizers, i.e., the preferences over $\triangle(S)$ are representable by a utility function $U_i : \triangle(S) \rightarrow R$ such that $U_i(\sigma) = \int_S U_i(s)\, d\sigma(s)$.

The relation $R^C(G) \subset D^{SM}(G)$ is rather obvious and it can be established under weaker assumptions than EU-maximizing players. Consider an extension of monotonicity with respect to SSD to mixed strategies such that $U_i(\sigma_i, \sigma_{-i}) > U(t_i, \sigma_{-i})$ whenever $U_i(\sigma_i, s_{-i}) > U(t_i, s_{-i})$ for all $s_{-i} \in S_{-i}$. If there exist a $\sigma_i \in \triangle(S_i)$ and a $s_i \in S_i$ such that $U(\sigma_i, s_{-i}) > U_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ then monotonicity with respect to SSD extended to mixed strategies implies $U(\sigma_i, \sigma_{-i}) > U_i(s_i, \sigma_{-i})$ for all $\sigma_{-i} \in \triangle(S_{-i})$. For a weakly gambling averse player there exists for every $\sigma_{-i} \in \triangle(S_{-i})$ some strategy $t_i \in S_i$ such that $U(t_i, \sigma_{-i}) > U_i(s_i, \sigma_{-i})$, i.e., s_i can not be a best response against any belief σ_{-i}.

The converse relation $D^{SM}(G) \subset R^C(G)$ is not so obvious. Pearce (1984) had proved $D^{SM}(G) \subset R^C(G)$ (Lemma 3) for finite two player-games via the existence of a saddlepoint in zero-sum games with mixed-strategy spaces (with the extension to any finite number of players easily at hand; compare also Lemma 3.2.1. and 3.2.2. for bimatrix games in van Damme 1991). Fudenberg and Tirole (1996) suggest that a direct application of the Separating Hyperplane Theorem for finite vector-spaces may offer a shortcut of Pearce's proof, and such a shortcut is indeed provided by the 'First fundamental theorem' in Berge (1997). The following observation restates the well-known equivalence result $D^{SM}(G) =$

$R^C(G)$ for strategy sets that are compact subsets of R^n.

Observation: *Given a game G such that*

(A1) Each player is an EU-maximizer.

(A2) S is a compact subset of R^n.

(A3) Each $U_i(s)$ is continuous with respect to s.

Then the strong dominance solution of G against mixed strategies coincides with the set of correlated rationalizable strategies of G, i.e., $D^{SM}(G) = R^C(G)$.

Besides the four different definitions of dominance criteria, mentioned above, Börgers (1993) proposes another dominance criterion for games with finite strategy sets. A strategy $s_i \in S_i$ is dominated in Börgers' sense if and only if there exists for every nonempty subset $T_{-i} \subset S_{-i}$ some strategy t_i such that $U(t_i, s_{-i}) \geq U(s_i, s_{-i})$ for all $s_{-i} \in T_{-i}$ and $U(t_i, s_{-i}) > U(s_i, s_{-i})$ for some $s_{-i} \in T_{-i}$. Let $D^B(G)$ denote the set of pure strategies that survive iterated elimination according to Börgers' dominance criterion, then it can be shown that $D^{WP}(G) \subset D^B(G) \subset D^{SP}(G)$ with strict inclusions for some games, i.e., Börgers dominance solution concept is something intermediate between the weak and the strong dominance solution against pure strategies.

Similarly to the definition of monotonicity with respect to SSD let us say that i's preferences satisfy monotonicity with respect to weak stochastic dominance (WSD) if $U_i(s_i, s_{-i}) \geq U(t_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ and $U_i(s_i, s_{-i}) > U(t_i, s_{-i})$ for all $s_{-i} \in T_{-i}$ implies $U_i(s_i, \sigma_{-i}) > U(t_i, \sigma_{-i})$ for all σ_{-i} with support on a finite set $T_{-i} \subset S_{-i}$. If a player's preferences satisfy monotonicity with respect to WSD any strategy that is dominated in Börgers' sense can not be a best response against some belief: if a belief on S_{-i} has support on T_{-i} a weakly dominating strategy t_i must be a strictly better response against this belief than s_i. Consequently, $R^C(G) \subset D^B(G)$.

In addition to his dominance criterion Börgers introduces an alternative definition of a 'rational' player. This definition differs from the preference–maximization assumption used in the rationalizability definitions of Bernheim and Pearce because it implies very strong restrictions on admissible (Expected Utility) preferences over $S_i \times \triangle(S_{-i})$. In particular, Börgers defines a strategy s_i as rational for player i if and only if there exists some belief σ_{-i} **and** some utility-number function $u_i : X \rightarrow R$ satisfying $u_i(x) \geq u_i(y) \Leftrightarrow x \succeq_i y$, such that

$$\sum_{s_{-i} \in S_{-i}} u_i(o(s_i, s_{-i})) * \sigma_{-i}(s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} u_i(o(t_i, s_{-i})) * \sigma_{-i}(s_{-i})$$

for all $t_i \in S_i$, with outcome-function $o : S \rightarrow X$. Consider the following payoff-(not utility!) matrix of player A

	$b1$	$b2$
$a1$	100	0
$a2$	ε	ε
$a3$	0	100

The strategy $a2$ is 'rational for player A' in Börgers sense for $\varepsilon > 0$, because $u(\varepsilon)$ can be chosen arbitrarily large as long as the original ranking of deterministic outcomes is not violated, i.e., $u(\varepsilon) < u(100)$. Börgers shows then for EU-maximizers that a strategy is dominated in his sense if and only if it is not rational according to his definition.

Based on Börgers' rationality definition we could define a set of rationalizable strategies in Börgers' sense, $R^B(G)$, and obtain $R^B(G) = D^B(G)$ for games with finite strategy sets and EU-maximizers. This equivalence result should not be mixed up with the seemingly similar result of Pearce $D^{SM}(G) = R^C(G)$, stated in the above observation, or with the equivalence results that will be derived in the remainder of this chapter. In the rationalizability definitions of Bernheim (1984) and Pearce (1984) it is assumed that each player's preferences over $S_i \times \triangle(S_{-i})$ are common knowledge, whereas it does not actually matter whether the players are EU-maximizers, or not.

In contrast, Börger's concept of rationalizability seems to be appropriate for strategic situations for which only the ranking of deterministic outcomes, plus the fact that all players are EU-maximizers, is common knowledge. Besides the uncertainty about the opponents' strategy choices which is also resolved by probabilistic beliefs there exists uncertainty about the players preferences over $S_i \times \triangle(S_{-i})$, except for the subset $S_i \times S_{-i}$. Let us denote by EU_i the set of all EU-preference representations $U_i : S_i \times \triangle(S_{-i}) \to R^+$ of player i such that the utility numbers satisfy $u_i(x) \geq u_i(y) \Leftrightarrow x \succeq_i y$. If this uncertainty about particular EU-preferences is resolved via point-beliefs we obtain Börger's rationalizability definition.

5.4 Results: Quasiconcave Utility

The results of this section are restricted to individual strategy sets that are compact intervals of the real line. Quasiconcavity of a player's utility function with respect to his own strategies guarantees basically the equivalence of point-rationalizable and of correlated rationalizable strategies (proposition 1). A strengthening of quasiconcavity to strict quasiconcavity implies the equivalence of the weak and of the strong dominance solution against pure strategies. Under the additional assumption of weakly gambling averse players a combination of this finding with proposition 1 and with Moulin's Lemma 2 establishes general equivalence of dominance- and rationalizability concepts (proposition 2).

Proposition 1: *Given a game G such that*

(A1) Each U_i satisfies monotonicity with respect to SSD.

(A2) The strategy sets S_i are convex and compact subsets of R .

(A3) Each U_i is continuous with respect to s.

(A4) Each U_i is quasiconcave with respect to s_i, i.e., for all $s_{-i} \in S_{-i}$

$$\begin{aligned} U_i\left(s_i, s_{-i}\right) &> U_i\left(t_i, s_{-i}\right) \Rightarrow U_i\left(as_i + (1-a)\, t_i, s_{-i}\right) > U_i\left(t_i, s_{-i}\right) \\ U_i\left(s_i, s_{-i}\right) &= U_i\left(t_i, s_{-i}\right) \Rightarrow U_i\left(as_i + (1-a)\, t_i, s_{-i}\right) \geq U_i\left(t_i, s_{-i}\right) \end{aligned}$$

for $a \in (0,1)$.

Then $\lambda^k(S) = \nu^k(S) = \mu^k(S)$ for $k \geq 0$. In particular $P(G) = R(G) = R^C(G)$.

Proposition 2: *Given a game G such that the assumptions (A1)-(A3) of proposition 1 are satisfied. If each player is weakly gambling averse and if each U_i is* ***strictly*** *quasiconcave with respect to s_i, i.e., for all $s_{-i} \in S_{-i}$*

$$\begin{aligned} U_i\left(s_i, s_{-i}\right) &> U_i\left(t_i, s_{-i}\right) \Rightarrow U_i\left(as_i + (1-a)\, t_i, s_{-i}\right) > U_i\left(t_i, s_{-i}\right) \\ U_i\left(s_i, s_{-i}\right) &= U_i\left(t_i, s_{-i}\right) \Rightarrow U_i\left(as_i + (1-a)\, t_i, s_{-i}\right) > U_i\left(t_i, s_{-i}\right) \end{aligned}$$

for $a \in (0,1)$ *and* $s_i \neq t_i$*, then*

$$\lambda^k(S) = \nu^k(S) = \mu^k(S) = \delta^k(S) = \theta^k(S) = \varphi^k(S) = \psi^k(S)$$

for $k \geq 0$*. In particular*

$$P(G) = R(G) = R^C(G) = D^{WP}(G) = D^{SP}(G) = D^{SM}(G) = D^{WM}(G)$$

Observe that the equivalence of $D^{SM}(G)$ and $R^C(G)$ in proposition 2 is not implied by the above observation (i.e., the equivalence result of Pearce 1984) because proposition 2 does not require EU-maximizers. Mainly because weak gambling aversion implies $P(G) \subset D^{SM}(G)$ the assumptions of proposition 2 are strong enough to derive $D^{SM}(G) = R^C(G)$ without any reference to a separation theorem as in the proof of the observation.

Strict quasiconcavity was not necessary for obtaining $P(G) = R(G) = R^C(G)$ in proposition 1, whereas strict quasiconcavity is crucial for $D^{WP}(G) = P(G)$ in Moulin's Lemma 2 as well as for $D^{WP}(G) = D^{SP}(G)$ in proposition 2 of this chapter. This assumption of strict-quasiconcavity is rather restrictive, because, in contrast to quasiconcavity, it implies single-valued best-response correspondences.

5.5 Results: Monotonic Utility Differences

This section introduces increasing and decreasing differences of the utility functions U_i in s_{-i} as equivalence conditions. Because the results of this section refer to concepts of lattice theory and of supermodular functions recall at first the following definitions (for more comprehensive definitions see Topkis 1979; Vives 1990; Milgrom and Roberts 1990; Fudenberg and Tirole 1996):

1. Given a reflexive, transitive, and antisymmetric binary relation $\leq_L$ on a set S_i let $(S_i, \leq_L)$ denote a lattice, i.e., for all elements $s_i, t_i \in S_i$ there exist a supremum $s_i \vee t_i$ and an infimum $s_i \wedge t_i$ in S_i.

2. $(S_i, \leq_L)$ is a complete lattice if $\inf T \in S_i$ and $\sup T \in S_i$ for every non-empty subset $T \subset S_i$. Note: completeness of S_i implies the existence of exactly one "smallest" element $s_i \in S_i$ such that $s_i \leq_L s_i'$ for all $s_i' \in S_i$, and of exactly one "largest" element $t_i \in S_i$ such that $s_i' \leq_L t_i$ for all $s_i' \in S_i$.

3. If $(S_i, \leq_L)$ is a lattice for every $i \in I$ then $(S, \leq_L)$ denotes a lattice such that $s \leq_L t$ iff $s_i \leq_L t_i$ for all i.

4. Following Milgrom and Roberts U_i is said to be order-continuous on a complete lattice $T \subset S$ if for any chain C (=totally ordered subset of T) $\lim_{s \in C, s \downarrow \inf C} U_i(s) = U_i(\inf C)$ and $\lim_{s \in C, s \uparrow \sup C} U_i(s) = U_i(\sup C)$. Order upper semicontinuity is accordingly defined as $\limsup_{s \in C, s \downarrow \inf C} U_i(s) \leq U_i(\inf C)$ and $\limsup_{s \in C, s \uparrow \sup C} U_i(s) \leq U_i(\sup C)$, (where $\limsup_{s \in C, s \uparrow \sup C} U_i(s)$ stands for $\wedge_{s \in C, s \uparrow \sup} \vee_{t \in C, s \leq_L t} U_i(t)$).

5. $U_i(s)$ is supermodular on S_i if for all $s_i, t_i \in S_i$

$$U_i(s_i, s_{-i}) + U_i(t_i, s_{-i}) \leq U_i(s_i \wedge t_i, s_{-i}) + U_i(s_i \vee t_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

6. $U_i(s)$ has "increasing differences" if $U_i(s_i, s_{-i}) - U_i(t_i, s_{-i})$ is non-decreasing in s_{-i} for $t_i \leq_L s_i$, and it has "decreasing differences" if $U_i(s_i, s_{-i}) - U_i(t_i, s_{-i})$ is non-increasing in s_{-i} for $t_i \leq_L s_i$.

Lemma: *Given a game G such that*

(A1) Each U_i satisfies monotonicity with respect to SSD.

(A2) Each S_i is a complete lattice $(S_i, \leq_L)$.

(A3) Each U_i is order upper semi-continuous with respect to s_i for fixed s_{-i}, and bounded from above.

(A4) Each U_i is supermodular on S_i.

(A5) Each U_i has either increasing or decreasing utility differences.

Then the sets $\lambda^k(S)$, $\nu^k(S)$, $\mu^k(S)$, and $\theta^k(S)$ are complete lattices with

$$\begin{aligned} \sup \lambda^k(S) &= \sup \nu^k(S) = \sup \mu^k(S) = \sup \theta^k(S) \\ \inf \lambda^k(S) &= \inf \nu^k(S) = \inf \mu^k(S) = \inf \theta^k(S) \end{aligned}$$

for $k \geq 0$. Moreover, suppose

(A6) Each U_i is order-continuous with respect to s_{-i}.

Then $P(G)$, $R(G)$, $R^C(G)$, and $D^{SP}(G)$ are complete lattices with

$$\begin{aligned} \sup P(G) &= \sup R(G) = \sup R^C(G) = \sup D^{SP}(G) \\ \inf P(G) &= \inf R(G) = \inf R^C(G) = \inf D^{SP}(G) \end{aligned}$$

The proof of the lemma is very similar to the proofs of lemma 1 and of theorem 5 in Milgrom and Roberts (1990). The main difference results from the fact that the Lemma allows also for decreasing utility differences whereas Milgrom and Roberts' definition of a supermodular game requires increasing utility differences for every player. As a consequence of this generalization the strategies $\sup D^{SP}(G)$ and $\inf D^{SP}(G)$ are not necessarily Nash equilibria, (as they are for a supermodular game), whenever $P(G)$ is not single-valued (compare theorem 5 in Milgrom and Roberts 1992).

Notice that $\sup P(G) = \sup D^{SP}(G)$ and $\inf P(G) = \inf D^{SP}(G)$ does not imply $P(G) = D^{SP}(G)$ under the assumptions of the Lemma. Consider the following example of a symmetric two player game with the payoff-matrix of player A given as

	$b1$	$b2$	$b3$
$a1$	1	1	1
$a2$	0.7	0.7	0.7
$a3$	0	1	1

Let $a1 \leq_L a2 \leq_L a3$ and $b1 \leq_L b2 \leq_L b3$, and observe that the assumptions (A1)-(A5) of the Lemma are satisfied. The individual strategy $a2$ is not a best response against any point belief but $a2$ is not strongly dominated by any pure strategy either. It remains to identify additional conditions that assure $P(G) = D^{SP}(G)$. Since any point-rationalizable strategy must belong to $D^{SP}(G)$ we obtain immediately the following corollary:

Corollary: *If a game G satisfies the assumptions (A1)-(A6) of the Lemma then $P(G) = R(G) = R^C(G) = D^{SP}(G)$ whenever $s \in P(G)$ for all $s \in S$ with $\inf P(G) \leq_L s \leq_L P(G)$.*

By the corollary $D^{SP}(G)$ and $P(G)$ trivially coincide for single-valued $P(G)$. Furthermore, the assumption of weak gambling aversion implies non-emptiness of $D^{WM}(G)$, and the inclusions $D^{WM}(G) \subset D^{SM}(G) \subset D^{SP}(G)$ give the following result.

Proposition 3: *If a game G satisfies the assumptions (A1)-(A6) of the Lemma then*

$$P(G) = R(G) = R^C(G) = D^{SP}(G) = D^{WP}(G)$$

whenever $P(G)$ is single-valued. Moreover, if weak gambling aversion is satisfied then $P(G) = D^{WM}(G) = D^{SM}(G)$ for single-valued $P(G)$.

Proposition 4: *Given a game G such that*

(A1) Each U_i satisfies monotonicity with respect to SSD.

(A2) The strategy sets S_i are convex and compact subsets of R.

(A3) The individual best response correspondences are upper-hemicontinuous and the $f_i(s_{-i})$ are convex for all $s_{-i} \in S_{-i}$.

(A4) Each U_i has either increasing or decreasing differences.

Then $\lambda^k(S) = \nu^k(S) = \mu^k(S) = \theta^k(S)$ for $k \geq 0$. In particular $P(G) = R(G) = R^C(G) = D^{SP}(G)$.

Observe that the results of proposition 1 and of proposition 4 are independent of eachother. The assumptions (A1)-(A3) of proposition 4 are also satisfied for proposition 1 but quasiconcavity and monotone utility differences are clearly independent conditions, e.g., for twice continuously utility functions monotone utility differences conditions are characterized via properties of the second-order cross-derivatives whereas quasiconcavity can be guaranteed via properties of the first-order partial derivatives. However, observe that the assumption of quasiconcavity is a sufficient though not a necessary condition for a convex-valued best response correspondence as required in proposition 4.

5.6 Discussion

Proposition 3 provides an equivalence result for games with a wide range of strategy sets, however, the claim for a unique point-rationalizable strategy is rather strong, (for an overview of uniqueness results for point-rationalizability see chapter 2 of this thesis). In contrast, the results of proposition 1 and of proposition 4 are restricted to individual strategy sets that are compact intervals of the real line whereas the assumptions of quasiconcavity, respectively of monotone utility differences, are less restrictive than the claim for uniqueness.

We have already seen that proposition 3 can not be weakened to non-unique point-rationalizable strategies. Moreover, it is easy to find examples of finite games for which the sets of point-rationalizable and of correlated rationalizable strategies differ due to non-convexity of the strategy sets, even if the other assumptions of proposition 1 or of proposition 4 are satisfied.

In this section I sketch the technical argumentation behind the results of proposition 1 and proposition 4 in order to show another limitation for a possible generalization of this chapter's equivalence results: proposition 1 and proposition 4 can not be extended to more general strategy sets, e.g., compact intervals of R^n with $n \geq 2$.

The proofs of proposition 1 and of proposition 4 show that each best response against some non-degenerated belief is also a best response against some point-belief over the same support, i.e., $\lambda^k(S) = \mu^k(S)$ for all k. Quasiconcavity, respectively monotone utility differences, guarantee then $\sup \lambda^k(S) = \sup \mu^k(S)$ and $\inf \lambda^k(S) = \inf \mu^k(S)$. Under the assumption of quasiconcavity (proposition 1) this result is restricted to real-valued, convex individual strategy sets. Under the assumption of monotone utility differences it is satisfied for much more general strategy sets.

In order to obtain $\lambda^k(S) = \mu^k(S)$ the following condition is obviously sufficient: if $\inf \lambda^k(S) \leq_L s \leq_L \sup \lambda^k(S)$, for some $s \in S$, then $s \in \lambda^k(S)$. For real-valued individual strategy sets this condition is trivially satisfied for convex sets $\lambda_i^k(S)$. In the proofs of proposition 1 and of proposition 4 convexity of $\lambda_i^k(S)$ is established via upper-hemicontinuity of the best response correspondence f_i plus convexity of $f_i(s_{-i})$ for all $s_{-i} \in S_{-i}$. Particularly simple would be a proof for single-valued upper-hemicontinuous best response correspondences: convexity of $\lambda_i^k(S)$ is immediately implied by the Intermediate Value Theorem, i.e., the continuous function f_i assumes every value between $\sup \lambda_i^k(S)$ and $\inf \lambda_i^k(S)$ since $\lambda_{-i}^{k-1}(S)$ is a connected set.

Unfortunately, this argumentation is not at hand for individual strategy sets that are not intervals of the real line: as a generalization of the Intermediate Value Theorem the set $\lambda_i^k(S)$ must be connected if $\lambda_{-i}^{k-1}(S)$ is connected. However, connectedness of $\lambda_i^k(S)$ is not sufficient to guarantee $s_i \in \lambda_i^k(S)$ for all $s_i \in S_i$, with $\inf \lambda_i^k(S) \leq_L s_i \leq_L \sup \lambda_i^k(S)$, whenever S_i is not anylonger an interval of the real line. To see the problem consider the following example:

Let $I = \{A, B\}$, $S_A = [0,1] \times [0,1]$, $S_B = [0,1]$, $U_A(s_A, s_B) = -(s_B - s_{A,1})^2 - \left(s_B^2 - s_{A,2}\right)^2$, and let the players be EU-maximizers. Notice, if we choose U_B properly, (e.g., $U_B(s_A, s_B) = 0$), we obtain a game that is supermodular in the sense of Milgrom and Roberts (compare Theorem 4 in Milgrom and Roberts 1990), and which satisfies therefore all the assumptions of the lemma. In particular, $\sup \lambda_A^1(S) = \sup \mu_A^1(S) = (1,1)$ and $\inf \lambda_A^1(S) = \inf \mu_A^1(S) = (0,0)$ and we have for all $s_A \in S_A$ that $\inf \lambda_A^k(S) \leq_L s_A \leq_L \sup \lambda_A^k(S)$. The best response function of player A against point-beliefs is given by $f_A(s_B) = (s_B, s_B^2)$, i.e., $\lambda_A^1(S) = \bigcup_{s_B \in [0,1]} \{(s_B, s_B^2)\}$. However, the best response against a belief σ_B, with $\sigma_B(0) = \sigma_B(1) = 0.5$, is given for an EU-maximizer by $f_A(\sigma_B) = (0.5, 0.5)$. Consequently, $f_A(\sigma_B) \notin \lambda_A^1(S)$ and therefore $\lambda_A^1(S) \neq \mu_A^1(S)$.

This game has comparably nice properties: it is supermodular (with increasing utility differences), the utility functions are quasiconcave with respect to individual strategies, the individual strategy sets are compact and convex subsets of R^n, I even assume EU-maximizers - nevertheless, we do neither obtain $\lambda^k(S) = \mu^k(S)$ for $k \geq 1$, nor $P(G) = R^C(G)$.

5.7 Concluding Remarks

In this chapter I have derived conditions which guarantee equivalence of point-rationalizability, rationalizability, and correlated rationalizability. Additionally, equivalence conditions for rationalizability concepts and for concepts of iterated elimination of dominated strategies have been obtained. The equivalence conditions of this chapter are closely related to similar conditions in Moulin (1984) and in Milgrom and Roberts (1990).

While the propositions of this chapter offer extensions and generalizations of the existing results they depend still on rather strong assumptions. The propositions 1,2, and

4 assume individual strategy sets that are compact intervals of the real line, and it is shown that neither the assumption of convexity nor the assumption of real-valued individual strategy sets can be relaxed for these propositions. Proposition 3 allows for more general strategy sets but it requires a unique point-rationalizable strategy; an assumption that can not be relaxed either. The findings of this chapter suggest that any further equivalence results have to make even stronger assumptions, most likely restricting the relevance of such results to very specific games.

5.8 Appendix: Proofs

Proof of the Observation: The relation $R^C(G) \subset D^{SM}(G)$ is already proved. By proving $D^{SM}(G) \subset R^C(G)$ I proceed along the lines of Berge's (1997) proof of the 'First Fundamental Theorem' (p. 200). Note at first: a strategy s_i is not strongly dominated by some mixed strategy σ_i if and only if the system of inequalities $U_i(\sigma_i, s_{-i}) > U_i(s_i, s_{-i})$, with $s_{-i} \in S_{-i}$, has no solution $\sigma_i \in \triangle(S_i)$. Define now the set

$$V(\sigma_i) = \left\{ (x(s_{-i}))_{s_{-i} \in S_{-i}} \in C[S_{-i}] \mid x(s_{-i}) \in R, x(s_{-i}) < U_i(\sigma_i, s_{-i}) \right\}$$

which collects continuous real functions $(x(s_{-i}))_{s_{-i} \in S_{-i}}$ on S_{-i} (=elements of $C[S_{-i}]$), and observe that by construction the function $(U_i(s_i, s_{-i}))_{s_{-i} \in S_{-i}}$ is not an element of $V = \bigcup_{\sigma_i \in \triangle(S_i)} V(\sigma_i)$ if s_i is not strongly dominated by some mixed strategy σ_i. Because the utility function is bounded the set V is some non-empty open set in the topology on $C[S_{-i}]$ generated by the supremums-norm, and it contains therefore by lemma 4.43 in Aliprantis and Border (1994) some 'internal point' as a prerequisite for the application of the Basic Separating Hyperplane Theorem (Theorem 4.42 in Aliprantis and Border 1994).

Furthermore, since V is clearly convex for EU-maximizers there exists by the Basic Separating Hyperplane Theorem (i.e., Hahn- Banach Theorem p. 157 in Berge 1997) a probability measure σ_{-i} such that

$$\int_{S_{-i}} x(s_{-i})\, d\sigma_{-i}(s_{-i}) \leq \int_{S_{-i}} U_i(s_i, s_{-i})\, d\sigma_{-i}(s_{-i})$$

for all $(x(s_{-i}))_{s_{-i} \in S_{-i}} \in V$, (the direction of the inequality results from the fact that the $x(s_{-i})$ can be chosen arbitrarily small). Let $x(s_{-i}) = U_i(\sigma_i, s_{-i}) - \varepsilon(s_{-i})$ and observe that

$$\int_{S_{-i}} U_i(\sigma_i, s_{-i}) - \varepsilon(s_{-i})\, d\sigma_{-i}(s_{-i}) \leq \int_{S_{-i}} U_i(s_i, s_{-i})\, d\sigma_{-i}(s_{-i})$$

holds for all $\varepsilon(s_{-i}) > 0$ which gives

$$\int_{S_{-i}} U_i(\sigma_i, s_{-i})\, d\sigma_{-i}(s_{-i}) \leq \int_{S_{-i}} U_i(s_i, s_{-i})\, d\sigma_{-i}(s_{-i})$$

for all $\sigma_i \in \triangle(S_i)$. But for an EU-maximizer this last inequality is equivalent to $U_i(\sigma_i, \sigma_{-i}) \leq U_i(s_i, \sigma_{-i})$, for all $\sigma_i \in \triangle(S_i)$. Consequently, if s_i is not strongly dominated by some mixed strategy $\sigma_i \in \triangle(S_i)$ then s_i must be a best response against the belief σ_{-i}. Finally, observe that σ_{-i} is an element of $\triangle(S_{-i})$ but not necessarily of $\triangle^I(S_{-i})$. This proves $D^{SM}(G) = R^C(G)$.$\square$

Proof of proposition 1: The proof proceeds in two steps.

step 1.) I show that under the assumptions (A2) and (A4a) of proposition 1 the sets $\lambda_i^k(S)$ are compact and convex subsets of R for all $k \geq 0$ and all i. Observe at first that f_i is upper-hemicontinuous by Berge's Maximum Theorem. Since S is compact and f_i is upper-hemicontinuous the set $\lambda_i^1(S)$ is compact and because this is true for all i the sets

$\lambda_i^k(S)$ are compact for all $k \geq 1$. It remains to show convexity of

$$\lambda_i^k(S) = \bigcup_{s_{-i} \in \lambda_{-i}^{k-1}(S)} f_i(s_{-i})$$

Suppose on the contrary that $\lambda_i^k(S)$ is not convex, then there must exist some $s_i, t_i \in \lambda_i^k(S)$ such that for some $a \in (0,1)$ we have $as_i + (1-a)t_i \notin \lambda_i^k(S)$. Since $f_i(s_{-i})$ is convex by quasiconcavity of $U_i(s)$ it follows $s_i \in f_i(s_{-i})$ and $t_i \in f_i(t_{-i})$ with $s_{-i} \neq t_{-i}$. By compactness of $f_i(r_{-i})$ for all $r_{-i} \in S_{-i}$ there must exist some $b \in (0,1)$ such that either

$$\begin{aligned} \min f_i(bs_{-i} + (1-b)t_{-i}) &> as_i + (1-a)t_i \text{ and} \\ \max f_i((b+\varepsilon)s_{-i} + (1-(b+\varepsilon))t_{-i}) &< as_i + (1-a)t_i \end{aligned}$$

for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$, or

$$\begin{aligned} \max f_i(bs_{-i} + (1-b)t_{-i}) &< as_i + (1-a)t_i \\ \min f_i((b+\varepsilon)s_{-i} + (1-(b+\varepsilon))t_{-i}) &> as_i + (1-a)t_i \end{aligned}$$

In either case upper-hemicontinuity is violated because there exists an open set V such that $f_i(bs_{-i} + (1-b)t_{-i}) \subset V$ and $as_i + (1-a)t_i \in V$ but **not**

$$f_i((b+\varepsilon)s_{-i} + (1-(b+\varepsilon))t_{-i}) \subset V$$

for $\varepsilon \in (0, \varepsilon_0)$. Hence, for no open neighborhood O around $bs_{-i} + (1-b)t_{-i}$ do we have $f_i(r_{-i}) \subset V$ for all $r_{-i} \in O$ as claimed by upper-hemicontinuity.

step 2.) Since every set $\lambda_i^k(S)$ is a compact subset of R it contains a unique maximum, $\max \lambda_i^k(S)$, and a unique minimum, $\min \lambda_i^k(S)$. Furthermore, by convexity of $\lambda_i^k(S) \subset R$ we have $s_i \in \lambda_i^k(S)$ for all s_i with $\min \lambda_i^k(S) \leq s_i \leq \max \lambda_i^k(S)$. Hence,

$\lambda^k(S) = \mu^k(S)$ iff there does not exist some $s_i \in \mu_i^k(S)$ such that $s_i < \min \lambda_i^k(S)$ or $\max \lambda_i^k(S) < s_i$. The max-case: Suppose there exists for some i a mixed belief σ_{-i}, with $\sigma_{-i} \in \triangle\left(\mu_{-i}^{k-1}(S)\right)$, and a strategy s_i such that $s_i \in f_i(\sigma_{-i})$ and $s_i > \max \lambda_i^k(S)$. Since $\max \lambda_i^k(S) = as_i + (1-a) f_i(s_{-i})$ for some $a \in [0,1)$, and $U_i(f_i(s_{-i}), s_{-i}) > U_i(s_i, s_{-i})$ for all s_{-i}, (by assumption s_i is not a best response against any pure strategy profile s_{-i}), it follows from quasiconcavity (A3) that $U_i\left(\max \lambda_i^k(S), s_{-i}\right) > U_i(s_i, s_{-i})$ for all s_{-i}. By monotonicity with respect to SSD $s_i \notin f_i(\sigma_{-i})$. For the min-case: $U_i\left(\min \lambda_i^k(S), s_{-i}\right) > U_i(s_i, s_{-i})$ for all s_{-i} in case $s_i < \min \lambda_i^k(S)$.□

Proof of proposition 2: Suppose the assumptions of proposition 2 are satisfied.

step 1.) $P(G) = R(G) = R^C(G)$ by proposition 1.

step 2.) $D^{WP}(G) = P(G)$ by Moulin's Lemma 2.

step 3.) $D^{WP}(G) = D^{SP}(G)$. Obviously, $D^{WP}(G) \subset D^{SP}(G)$. I prove now $\delta^k(S) = \theta^k(S)$, for $k \geq 0$. Note $\delta^0(S) = \theta^0(S)$, and as induction assumption suppose $\delta^{k-1}(S) = \theta^{k-1}(S)$. It is true that $\delta^k(S) \neq \theta^k(S)$ if and only if there exist some $s_i, t_i \in \delta_i^{k-1}(S)$ such that $U(t_i, s_{-i}) \geq U(s_i, s_{-i})$ for all $s_{-i} \in \delta_{-i}^{k-1}(S)$, and $U(t_i, s_{-i}) > U(s_i, s_{-i})$ for some $s_{-i} \in \delta_{-i}^{k-1}(S)$, but there does not exist a $r_i \in \delta_i^{k-1}(S)$ such that $U(r_i, s_{-i}) > U(s_i, s_{-i})$ for all $s_{-i} \in \delta_{-i}^{k-1}(S)$. By strict-quasiconcavity it follows for an arbitrary $a \in (0,1)$ that $U_i(at_i + (1-a)s_i, s_{-i}) > U_i(s_i, s_{-i})$ for all $s_{-i} \in \delta_{-i}^{k-1}(S)$. Since $\delta_i^{k-1}(S) = \lambda_i^{k-1}(S)$ for all $k \geq 1$, $\delta_i^{k-1}(S)$ is convex for all $k \geq 1$ by the proof of proposition 1. Consequently, there exists some $r_i \in \delta_i^{k-1}(S)$, with $r_i = at_i + (1-a)s_i$ for some $a \in (0,1)$, such that $U(r_i, s_{-i}) > U(s_i, s_{-i})$ for all $s_{-i} \in \delta_{-i}^{k-1}(S)$.

step 4.) $D^{SP}(G) = D^{WM}(G)$. Obviously, $D^{WM}(G) \subset D^{SP}(G)$. Furthermore, notice that $P(G) \subset D^{WM}(G)$: by strict quasiconcavity each $s_i \in \lambda_i^k(S)$ is a unique best response against some $s_{-i} \in \lambda_{-i}^{k-1}(S)$, and by weak randomization aversion there can not exist some mixed strategy σ_i, with $\sigma_i \neq s_i$, such that $U(\sigma_i, s_{-i}) \geq U_i(s_i, s_{-i})$ for all $s_{-i} \in \lambda_{-i}^{k-1}(S)$. Consequently, $D^{SP}(G) \subset D^{WM}(G)$ by step 1.) - step 3.).□

Proof of the Lemma: I prove $\sup \lambda^k(S) = \sup \theta^k(S)$ and $\inf \lambda^k(S) = \inf \theta^k(S)$ for $k \geq 0$, which implies $\sup \lambda^k(S) = \sup \mu^k(S)$ and $\inf \lambda^k(S) = \inf \mu^k(S)$ under (A1). (Note: the existence of $\sup \theta^k(S)$ and $\inf \theta^k(S)$ is guaranteed by Theorem 1 and Theorem 2 in Milgrom and Roberts, 1990). Consider a player i with decreasing utility differences. Given an interval $[s_{-i}, t_{-i}]$ with $s_{-i} \leq_L t_{-i}$ let $\theta_i[s_{-i}, t_{-i}]$ denote the set of undominated strategies and let $\lambda_i[s_{-i}, t_{-i}]$ denote the set of best responses against point-beliefs over $[s_{-i}, t_{-i}]$. Let $\hat{s}_i = \sup f_i(s_{-i})$, $\check{s}_i = \inf f_i(t_{-i})$ where $\hat{s}_i \in f_i(s_{-i})$, $\check{s}_i \in f_i(t_{-i})$ exist by (A2), (A3), and (A4). Observe that any r_i with $r_i \not\leq_L \hat{s}_i$ is strongly dominated by the strategy $\hat{s}_i \wedge r_i$: for all $x_{-i} \in [s_{-i}, t_{-i}]$

$$\begin{aligned} U_i(r_i, x_{-i}) - U_i(\hat{s}_i \wedge r_i, x_{-i}) &\leq U_i(r_i, s_{-i}) - U_i(\hat{s}_i \wedge r_i, s_{-i}) \text{ by (A5)} \\ &\leq U_i(\hat{s}_i \vee r_i, s_{-i}) - U_i(\hat{s}_i, s_{-i}) \text{ by (A4)} \\ &< 0 \end{aligned}$$

where the last inequality follows from $\hat{s}_i \in f_i(s_{-i})$ and $\hat{s}_i <_L \hat{s}_i \vee r_i$, i.e., $\hat{s}_i \vee r_i \notin f_i(s_{-i})$. This proves $r_i \leq_L \hat{s}_i$ for any $r_i \in \theta_i[s_{-i}, t_{-i}]$. Accordingly, it can be shown that any strategy r_i with $\check{s}_i \not\leq_L r_i$ is dominated by a strategy $\check{s}_i \vee r_i$. Consequently, $\check{s}_i \leq_L r_i$ for any $r_i \in \theta_i[s_{-i}, t_{-i}]$. Since $\lambda_i[s_{-i}, t_{-i}] \subset \theta_i[s_{-i}, t_{-i}]$ it follows $\sup \lambda_i[s_{-i}, t_{-i}] = \sup \theta_i[s_{-i}, t_{-i}]$

and $\inf \lambda_i [s_{-i}, t_{-i}] = \inf \theta_i [s_{-i}, t_{-i}]$ for any interval $[s_{-i}, t_{-i}]$ with $s_{-i} \leq_L t_{-i}$. Nearly the same argument had been applied in the proof of lemma 1 in Milgrom and Roberts for deriving $\inf \lambda_i [s_{-i}, t_{-i}] = \inf \theta_i [s_{-i}, t_{-i}]$ for a player with increasing utility differences. By assumption S_{-i} is a complete lattice and if we let $[s_{-i}, t_{-i}] = S_{-i}$ for all i then $\sup \lambda^1 (S) = \sup \theta^1 (S)$ and $\inf \lambda^1 (S) = \inf \theta^1 (S)$ if each player has either decreasing or increasing utility differences. Furthermore, observe that $\lambda^1 (S)$ is a complete lattice itself and by induction the same argument yields $\sup \lambda^k (S) = \sup \theta^k (S)$ and $\inf \lambda^k (S) = \inf \theta^k (S)$ for $k \geq 0$. This proves the first part of the Lemma.

Let us now turn to the additional assumption (A6) of order-continuity. Since the sequences $\{\inf \lambda^k (S)\}_{k\geq 0}$, $\{\sup \lambda^k (S)\}_{k\geq 0}$ are monotonically non-decreasing, respectively non-increasing, lattice-completeness of S implies the existence of order-limits $\lim_{k\to\infty} \inf \lambda^k (S) = \lim_{k\to\infty} \inf \theta^k (S) = \check{s}$ and $\lim_{k\to\infty} \sup \lambda^k (S) = \lim_{k\to\infty} \sup \theta^k (S) = \hat{s}$, such that $P(G), D^{SP}(G) \subset [\check{s}, \hat{s}]$. Consider a player i with decreasing utility differences, i.e., $\sup \lambda_i^k (S) \in f_i \left(\inf \lambda_{-i}^{k-1} (S)\right)$ for all k. Observe that $\check{s}_i$ must be a best response against $\hat{s}_{-i}$: if it was not then $U_i (s_i, \hat{s}_{-i}) - U_i (\hat{s}_i, \hat{s}_{-i}) < 0$ for some $s_i \neq \hat{s}_i$, which implies by order-continuity the existence of some number k such that

$$U_i \left(s_i, \inf \lambda_{-i}^{k-1} (S)\right) - U_i \left(\sup \lambda_i^k (S), \inf \lambda_{-i}^{k-1} (S)\right) < 0$$

,a contradiction. Analogously, $\hat{s}_i$ must be a best response against $\check{s}_{-i}$. For a player with increasing utility differences we have accordingly $\hat{s}_i \in f_i (\hat{s}_{-i})$ and $\check{s}_i \in f_i (\check{s}_{-i})$. Consequently, $\check{s}$ and $\hat{s}$ are themselves best-responses, as well as undominated, and we obtain $\inf P(G) = \inf D^{SP}(G) = \check{s}$ and $\sup P(G) = \sup D^{SP}(G) = \hat{s}$.$\square$

Proof of proposition 3: As shown in the proof of proposition 1 the assumptions (A2) and (A3) assure that the $\lambda^k(S)$ are non-empty and convex sets for all $k \geq 0$. Furthermore, the $\lambda^k(S)$ are complete lattices for all $k \geq 0$ under the natural ordering: $s \leq_L t$ iff $s_i \leq t_i$ for all i. Consequently, in view of the Lemma we obtain $\sup \lambda^k(S) = \sup \theta^k(S)$ and $\inf \lambda^k(S) = \inf \theta^k(S)$ for $k \geq 0$. Furthermore, convexity of $\lambda^k(S)$ implies $\lambda^k(S) = \theta^k(S)$ for $k \geq 1$. $\square$

Bibliography

[1] Aliprantis, D.C., and K. Border (1994), *Infinite Dimensional Analysis.* Springer-Verlag. Berlin, Heidelberg.

[2] Allais, M. (1953) "Le Comportement de l'Homme Rationnel devant le Risque: Critique des postulats et axiomes de l'École Americaine", *Econometrica* **21**, 503-546. Reprinted in English as part II in Allais, M., and O. Hagen [eds.] (1979), *Expected Utility Hypotheses and the Allais Paradox*, D. Reidel: Dordrecht etc.

[3] Basu, K. (1992), "A Characterization of the Class of Rationalizable Equilibria of Oligopoly Games", *Economics Letters* **40**, 187-191.

[4] Battigalli, P., and M. Siniscalchi (2000), "Rationalizable Bidding in General First-Price Auctions". *mimeo.*

[5] Berge, C. (1997), *Topological Spaces.* Dover Publications, Inc., Mineola, New York.

[6] Bernheim, B. Douglas (1984), "Rationalizable Strategic Behavior", *Econometrica* **52**, 1007-1028.

[7] Bonsall, F.F. (1962), *Lectures On Some Fixed Point Theorems Of Functional Analysis.* Tata Institute of Fundamental Research, Bombay.

[8] Börgers, T. (1993), "Pure Strategy Dominance", *Econometrica* **61**, 423-430.

[9] Börgers, T., and M.C.W. Janssen (1995), "On the Dominance Solvability of Large Cournot Games", *Games and Economic Behavior* **8**, 297-321.

[10] Brandenburger, A., and E. Dekel (1987), "Rationalizability and Correlated Equilibria", *Econometrica* **55**, 1391-1402.

[11] Cohen, M. (1992), "Security Level, Potential Level, Expected Utility: A Three-Criteria Decision Model under Risk", *Theory and Decision* **33**, 101-104.

[12] Crawford, V. P. (1990), "Equilibrium without Independence", *Journal of Economic Theory* **50**, 127-154.

[13] Dekel, E., and A. Wollinsky (2001), "Rationalizable outcomes of large independent private-value first-price discrete auctions". *mimeo.*

[14] Diecidue, E., Schmidt, U., and P. Wakker (2001), "The Gambling Effect Reconsidered", mimeo.

[15] Epstein (1997), "Preference, Rationalizability and Equilibrium", *Journal Of Economic Theory* **73**, 1-29.

[16] Essid, S. (1997), "Choice under Risk with Certainty and Potential Effects: A General Axiomatic Model", *Mahematical Social Sciences* **34**, 223-247.

[17] Fudenberg, D., and J. Tirole (1996), *Game Theory*. The MIT Press, Cambridge, Massachusetts.

[18] Gilboa, I. (1988), "A Combination of Expected Utility and Maxmin Decision Criteria", *Journal of Mathematical Psychology* **32**, 405-420.

[19] Guesnerie, R. (2002), "Anchoring Economic Predictions in Common Knowledge", *Econometrica* **70**, 439-480.

[20] Hausner (1954), "Multidimensional Utilities", in Thrall, R. M., Coombs, C. H., and R. L. Davids [eds.], *Decision Process*, John Wiley & Sons: New York etc., 167-180.

[21] Heuser, H. (1998), *Lehrbuch der Analysis, Teil 2*. B. G. Teubner, Stuttgart.

[22] Karni, E., and D. Schmeidler (1991), "Utility Theory with Uncertainty" in: Hildenbrand, W. and H. Sonnenschein [eds.], *Handbook of Mathematical Economics*, Vol. IV, North-Holland: Amsterdam etc., 1763-1831.

[23] Kahneman, D., and A. Tversky (1979), "Prospect Theory: An Analysis of Decision under Risk", *Econometrica* **47**, 263-291.

[24] Jaffray, J. Y. (1988), "Choice under Risk and the Security Factor: An Axiomatic Model", *Theory and Decision* **24**, 169-200.

[25] Lopes, L. L. (1987), "Between Hope and Fear: The Psychology of Risk" in: Berkowitz, L. [ed.], *Advances In Experimental Social Psychology*, Vol. **20**, New York: Academic Press, 255-295.

[26] Milgrom, P., and J. Roberts (1990), "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementaries", *Econometrica* **58**, 1255-1277.

[27] Moulin, H. (1984), "Dominance Solvability and Cournot Stability", *Mathematical Social Sciences* **7**, 83-102.

[28] Myerson, R. B. (1978), "Refinements of the Nash Equilibrium Concept", *International Journal of Game Theory* **7**, 73-80.

[29] Nash, J. F. (1950a), "Equilibrium points in N-Person Games", *Proceedings of the National Academy of Sciences* **36**. 48-49. Reprinted as chapter 5 in H.W. Kuhn, and S. Nasar [eds.] (2002), *The Essential John Nash.* Princeton University Press, Princeton and Oxford.

[30] Nash, J. F. (1950b), "Non-Cooperative Games.", *Ph.D. dissertation Princeton University.* Reprinted (facsimile) as chapter 6 in H.W. Kuhn, and S. Nasar [eds.] (2002), *The Essential John Nash.* Princeton University Press, Princeton and Oxford.

[31] v. Neuman, J. (1928) , "Zur Theorie der Gesellschaftsspiele", *Mathematische Annalen* **100**, 295-320.

[32] v. Neuman, J., and O. Morgenstern (1947), *Theory of Games and Economic Behavior*, Princeton University Press, 2nd edn., Princeton.

[33] Pearce, D. G. (1984), "Rationalizable Strategic Behavior and the Problem of Perfection", *Econometrica* **52**, 1029-1050.

[34] Schmidt, U. (1998b), *Axiomatic Utility Theory under Risk*, Springer Verlag, Berlin Heidelberg.

[35] Schmidt, U., and A. Zimper (2003), "Security- and Potential Level Preferences with Threshold Values", *mimeo*.

[36] Selten, R. (1975), "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games", *International Journal of Game Theory* **4**, 25-55.

[37] Starmer, C. (2000), "Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk", *Journal of Economic Literature* **38**, 332-382.

[38] Tan, T., and S.R. Costa Da Werlang (1988), "On Aumann's Notion of Common Knowledge, an Alternative Approach", *Journal of Economic Theory 45*, 370-391.

[39] Topkis, D. (1979), "Equilibrium Points in Nonzero-Sum n-Person Submodular Games", *SIAM Journal of Control and Optimization* **17**, 773-787.

[40] van Damme, E. (1991), *Stability and Perfection of Nash Equilibria. Second edition.* Springer Verlag.

[41] Vassiliev, D. (2001), "Real and Abstract Analysis", *mimeo*.

[42] Vives, X. (1990), "Nash Equilibrium with Strategic Complementarities", *Journal of Mathematical Economics* **19**, 305-321.

Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich die Dissertation selbständig angefertigt und mich anderer als der in ihr angegebenen Hilfsmittel nicht bedient habe, insbesondere, dass aus anderen Schriften Entlehnungen, soweit sie in der Dissertation nicht ausdrücklich als solche gekennzeichnet und mit Quellenangaben versehen sind, nicht stattgefunden haben.

Berlin, den 28.03.2003 Alexander Zimper

Curriculum Vitae

Name	Alexander Zimper
Geburtsdatum	21. Juni 1970
Geburtsort	Berlin

Schule

1977 - 1987	19. Polytechnische Oberschule, Berlin
1987 – 1989	Erweiterte Polytechnische Oberschule „Friedrich List“, Berlin
1989	Abitur

Tätigkeiten

1989	Einberufung zur Nationalen Volksarmee der DDR
1989 – 1990	Anstellung als Pflegerische Hilfskraft in der Forensischen Psychiatrie, Klinikum Berlin-Buch

Studium

1990 – 1997	Studium der Philosophie, Betriebswirtschaftslehre und Volkswirtschaftslehre an der Humboldt Universität zu Berlin, Auslandsstudium an der Stockholmer Universität und an der Stockholmer Handelshochschule (1997)
1997	Diplom im Fach Volkswirtschaftslehre, Humboldt Universität zu Berlin

Promotion

1998 – 2003	Promotionsstudium im Rahmen des Graduiertenkollegs “Allokation auf Finanz- und Gütermärkten“ der Universität Mannheim, Forschungsaufenthalt am University College London (2001)
2002 – 2003	Wissenschaftlicher Mitarbeiter am Lehrstuhl für Finanzwissenschaft und Sozialpolitik, von Prof. Dr. Christian Seidl, Fakultät für Volkswirtschaftslehre, Christian-Albrechts-Universität zu Kiel